# Advanced optimal control: from reusable rocket landing to efficient training of neural networks

IPSA course 2025

Riccardo Bonalli - riccardo.bonalli@cnrs.fr

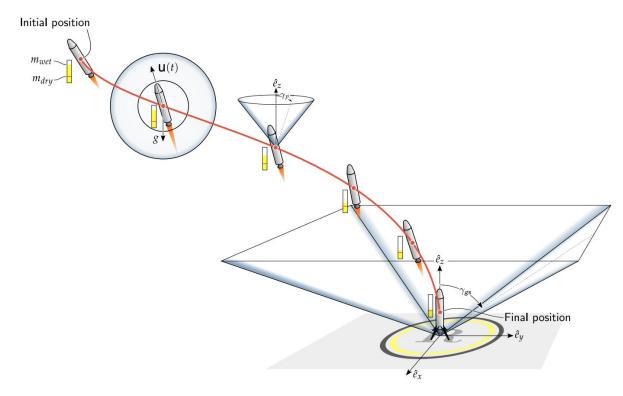
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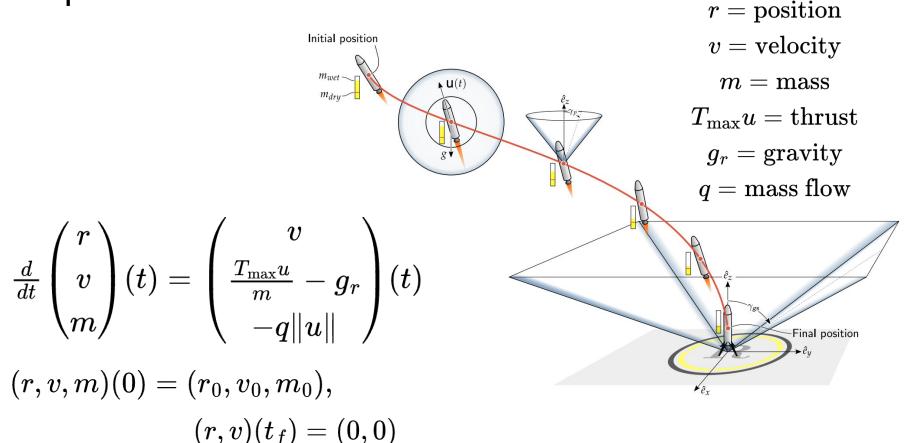


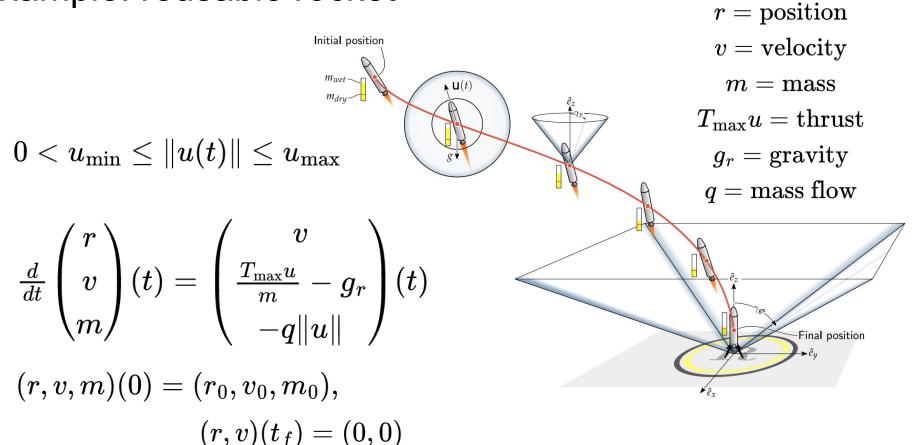
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$$\min_u - m(t_f)$$

$$0 < u_{\min} \leq \|u(t)\| \leq u_{\max}$$

$$rac{d}{dt}egin{pmatrix} r \ v \ m \end{pmatrix}(t) = egin{pmatrix} v \ rac{T_{ ext{max}u}}{m} - g_r \ -q \|u\| \end{pmatrix}(t)$$

 $egin{aligned} (r,v,m)(0) &= (r_0,v_0,m_0), \ &(r,v)(t_f) &= (0,0) \end{aligned}$ 

$$\min_u - m(t_f)$$

 $0 < u_{\min} \leq \|u(t)\| \leq u_{\max}$ 

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Dynamics + initial/final conditions  $\left\{ egin{array}{ll} x(t) \in \mathbb{R}^n, & t \in [0,t_f], \ \dot{x}(t) = f(t,x(t),u(t)), \ x(0) = x_0, & g(x(t_f)) = 0 \end{array} 
ight.$ 

$$\min_u \ -m(t_f)$$

 $0 < u_{\min} \leq \|u(t)\| \leq u_{\max}$ (Control) constraints  $u(t)\in U\subseteq \mathbb{R}^m, \ \ t\in [0,t_f]$  $rac{d}{dt}egin{pmatrix} r \ v \ m \end{pmatrix}(t) = egin{pmatrix} rac{v}{T_{ ext{max}u}} - g_r \ - a \|u\| \end{pmatrix}(t)$ Dynamics + initial/final conditions  $\left\{egin{aligned} x(t) \in \mathbb{R}^n, & t \in [0,t_f], \ \dot{x}(t) = f(t,x(t),u(t)), \ x(0) = x_0, & g(x(t_f)) = 0 \end{array}
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ight.$  $(r,v)(t_f) = (0,0)$  $(r, v, m)(0) = (r_0, v_0, m_0),$ 

## Example: reusable rocket Cost (functional) $\min_{u(\cdot)\in L^\infty} h(x(t_f)) + \int_0^{t_f} f^0(t,x(t),u(t)) \ dt$ $\min_u - m(t_f)$ — $0 < u_{\min} \leq \|u(t)\| \leq u_{\max}$ (Control) constraints $u(t)\in U\subseteq \mathbb{R}^m, \ \ t\in [0,t_f]$ $rac{d}{dt}egin{pmatrix} r \ v \ m \end{pmatrix}(t) = egin{pmatrix} rac{v}{T_{ ext{max}u}} - g_r \ - a \| u \| \end{pmatrix}(t)$ Dynamics + initial/final conditions $\left\{egin{aligned} x(t) \in \mathbb{R}^n, & t \in [0,t_f], \ \dot{x}(t) = f(t,x(t),u(t)), \ x(0) = x_0, & g(x(t_f)) = 0 \end{array} ight.$ $(r, v, m)(0) = (r_0, v_0, m_0),$ $(r,v)(t_f)=(0,0)$

This is an Optimal Control Problem (OCP)

Will learn theory and algorithms to efficiently solve OCP

$$\sum_{u(\cdot)\in L^\infty} h(x(t_f)) + \int_0^{t_f} f^0(t,x(t),u(t)) \; dt$$

(Control) constraints

$$u(t)\in U\subseteq \mathbb{R}^m, \;\;t\in [0,t_f]$$

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#### Cost (functional)

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#### **Key difficulties**

1. OCP is infinite dimensional (controls are in L infinity)

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This is an Optimal Control Problem (OCP)

Will learn theory and algorithms to efficiently solve OCP

#### **Key difficulties**

- 1. OCP is infinite dimensional (controls are in L infinity)
  - OCP is non-convex (dynamics and cost are non-convex)

 $\sum_{u(\cdot)\in L^\infty} h(x(t_f)) + \int_0^{t_f} f^0(t,x(t),u(t)) dt$ 

(Control) constraints

$$u(t)\in U\subseteq \mathbb{R}^m, \;\;t\in [0,t_f]$$

$$egin{aligned} & (x(t) \in \mathbb{R}^n, \ \ t \in [0,t_f], \ & \dot{x}(t) = f(t,x(t),u(t)), \ & x(0) = x_0, \ \ \ g(x(t_f)) = 0_{12} \end{aligned}$$

## **Course objectives**

1. Gain a strong foundation in modern optimal control techniques to develop efficient algorithms for controlling complex dynamical systems

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#### **Ultimate goal**

Get a comprehensive understanding of the key challenges faced by contemporary aerospace control engineers, addressing both theoretical and practical aspects

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- 5. Final presentation of the results (the SOLE evaluation step).

Group evaluation: please form groups of 2 / 3 people and <u>fill the Excel file on Moodle ASAP</u>! 20

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#### Lecture 1 - Existence of solutions to OCP

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 $\sum_{u(\cdot)\in L^\infty} h(x(t_f)) + \int_0^{t_f} f^0(t,x(t),u(t)) \; dt$ 

(Control) constraints $u(t)\in U\subseteq \mathbb{R}^m, \ \ t\in [0,t_f]$ 

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$$\begin{array}{ll} \text{Cost (functional)} & h: \mathbb{R}^n \to \mathbb{R} \\ \min_{u(\cdot) \in L^{\infty}} h(x(t_f)) + \int_0^{t_f} f^0(t, x(t), u(t)) \ dt & \longrightarrow & f^0: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \end{array}$$

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$$egin{aligned} & egin{aligned} & egin{aligned} & f: \mathbb{R}_+ imes \mathbb{R}^n imes \mathbb{R}^n o \mathbb{R}^n \ & g: \mathbb{R}^n o \mathbb{R}^\ell & {}_{24} \end{aligned}$$

$$egin{aligned} ext{Cost (functional)} & h: \mathbb{R}^n o \mathbb{R} \ & \mu: \mathbb{R}^n o \mathbb{R} \ & \mu: \mathbb{R}^n o \mathbb{R} \ & \mu: \mathbb{R}^n o \mathbb{R} \ & f^0: \mathbb{R}_+ imes \mathbb{R}^n imes \mathbb{R}^m o \mathbb{R} \end{aligned}$$

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$$u(t)\in U\subseteq \mathbb{R}^m, \ \ t\in [0,t_f]$$

Dynamics + initial/final conditions

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Key facts

- 1. The final time  $t_f$  may be free
- 2. All mappings are continuous...

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Cost (functional)  $\min_{u(\cdot)\in L^\infty} h(x(t_f)) + \int_0^{t_f} f^0(t,x(t),u(t)) \ dt \quad \longrightarrow \quad f^0: \mathbb{R}_+ imes \mathbb{R}^n imes \mathbb{R}^m o \mathbb{R}$ 

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Key facts

The final time  $t_f$  may be free 1.

 $h: \mathbb{R}^n \to \mathbb{R}$ 

All mappings are continuous... 2. except for controls!

$$egin{array}{lll} {
m 3.} & u:[0,t_f]
ightarrow U \ {
m such that} & u(\cdot)\in L^\infty([0,t_f],U) \end{array}$$

## Today's detailed schedule

- 1. Revisit the concept of  $L^{\infty}([0, t_f], U)$ .
- 2. Revisit Ordinary Differential Equations (ODE): existence and uniqueness of solutions given non smooth controls.
- 3. Conditions for the existence of solutions to OCP and application to the reusable rocket landing problem.

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### Quick reminders on Lp spaces - Theory

In this course we will often deal with:

$$L^{\infty}([0,t_f],U) riangleq \{u: [0,t_f] 
ightarrow U \subseteq \mathbb{R}^m ext{ s.t.} \ u(\cdot) ext{ is } egin{array}{ll} measurable & ext{ and } & \|u(\cdot)\|_{L^{\infty}} < \infty \} \end{array}$$

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In this course we will often deal with:

 $L^{\infty}([0, t_f], U) \triangleq \{u : [0, t_f] \to U \subseteq \mathbb{R}^m \text{ s.t.}$  $u(\cdot) \text{ is } \textit{measurable} \text{ and } ||u(\cdot)||_{L^{\infty}} < \infty\}$ 1. "Basically", integrals like  $\int_0^t \varphi(s, u(s)) \, ds$ are well-defined for every  $t \in [0, t_f]$  and  $\varphi \in C^0([0, t_f] \times \mathbb{R}^m, \mathbb{R}^\ell)$ 

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are well-defined for every  $t \in [0, t_{f}]$  and  $\varphi \in C^{0}([0, t_{f}] \times \mathbb{R}^{m}, \mathbb{R}^{\ell})$ 
2. The "supremum" norm  $\|u(\cdot)\| \triangleq \sup_{t \in [0, t_{f}]} \|u(t)\|$  is well-defined and finite.

### Quick reminders on Lp spaces - Example

In this course we will often deal with:

$$egin{aligned} L^\infty([0,t_f],U)&\triangleq \{u:[0,t_f] o U\subseteq \mathbb{R}^m ext{ s.t.}\ &u(\cdot) ext{ is } egin{aligned} measurable & ext{ and } & \|u(\cdot)\|_{L^\infty}<\infty \} \end{aligned}$$

Example: bang-bang controls

$$m=1, \hspace{0.2cm} U riangleq [-1,1]\subseteq \mathbb{R}, \hspace{0.2cm} t_{sw}\in [0,t_f],$$

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## Reminders on ODE - Theory From now on: $u(\cdot) \in L^\infty([0,t_f],U), \quad U \subseteq \mathbb{R}^m$ (it might hold $U = \mathbb{R}^m$ )

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In this course we will often deal with:

ODE 
$$egin{cases} \dot{x}(t) = f(t,x(t),u(t)), \ t\in [0,t_f], \ x(0) = x_0 \end{cases}$$
 where

$$f:\mathbb{R}_+ imes\mathbb{R}^n imes\mathbb{R}^m o\mathbb{R}^n$$

is "at least" continuous

From now on:  $u(\cdot)\in L^\infty([0,t_f],U), \quad U\subseteq \mathbb{R}^m$  (it might hold  $U=\mathbb{R}^m$  )

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1. A solution to ODE is a <u>continuous</u> curve  $x: [0, t_f] \to \mathbb{R}^n$  such that:  $x(0) = x_0$  and  $\dot{x}(t) = f(t, x(t), u(t)), \text{ for } t \in [0, t_f]$ 

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2. We need to answer the following question:

Under what assumptions on f does a solution to ODE <u>exist uniquely</u>?

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3. <u>Assumptions ODE</u> - There exist  $\bar{x} \in \mathbb{R}^n$  and a  $C^0$  function  $L(\cdot) > 0$  such that:

From now on:  $u(\cdot)\in L^\infty([0,t_f],U), \quad U\subseteq \mathbb{R}^m$  (it might hold  $\ U=\mathbb{R}^m$  )

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- $B. \hspace{0.2cm} \|f(t,x,u)-f(t,y,u)\|\leq L(\|u\|)\|x-y\|, \hspace{0.2cm} (t,u)\in [0,t_f] imes U, \hspace{0.2cm} x,y\in \mathbb{R}^n \hspace{0.2cm} (Lipschitzianity)$

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- 3. Assumptions ODE There exist  $\bar{x} \in \mathbb{R}^n$  and a  $C^0$  function  $L(\cdot) > 0$  such that:
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 $B. \quad \|f(t,x,u)-f(t,y,u)\| \leq L(\|u\|)\|x-y\|, \quad (t,u) \in [0,t_f] \times U, \ x,y \in \mathbb{R}^n \ (Lipschitzianity)$ 

Could be replaced with

open subset  $A \subseteq \mathbb{R}^n$ 

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- 4. <u>Theorem ODE</u> (Existence and uniqueness of solutions to ODE). Under <u>Assumptions ODE</u>, ODE has a unique solution  $x : [0, t_f] \to \mathbb{R}^n$ .

Could be replaced with

open subset  $A \subseteq \mathbb{R}^n$ 

Double integrator - Used in electronic circuit modeling

$$egin{cases} \dot{y}(t) = z(t)\ \dot{z}(t) = u_{t_{sw}}(t) & ext{where}\ t \in [0,t_f], \ x(0) = y(0) = 0 \end{cases}$$

$$egin{aligned} \mathbf{e} & u_{t_{sw}}(t) riangleq egin{cases} -1, & t \leq t_{sw} \ 1, & t > t_{sw} \end{aligned}, \quad t \in [0, t_f] \ 1, & t > t_{sw} \end{aligned}$$
 (clearly  $u_{t_{sw}} \in L^\infty([0, t_f], [-1, 1])$  )

Double integrator - Used in electronic circuit modeling

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 where  $u_{t_{sw}}(t) riangleq egin{cases} -1, & t \leq t_{sw}\ 1, & t > t_{sw}\ 1, & t > t_{sw} \end{cases}, \ t \in [0,t_f]\ ( ext{clearly} \ u_{t_{sw}} \in L^\infty([0,t_f],[-1,1])\ ) \end{cases}$ 

1. The dynamics f are:

$$n=2,m=1,~~x riangleq(y,z),~~f(t,x,u) riangleqinom{z}{u}$$

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 are:

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 $u$  Note that  $f$  must be defined for every  $u \in \mathbb{R}$  and not just at  $u_{t_{sw}}(\cdot)$  !!

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$$egin{aligned} \|f(t,0,u)\| &= |u| & ext{ and } \|f(t,x_1,u) - f(t,x_2,u)\| &= igg\|igg(z_1-z_2\ 0\ )igg\| \ &= |z_1-z_2| \leq \|x_1-x_2\| \end{aligned}$$

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$$\|f(t,0,u)\| = |u|$$
 and  $\|f(t,x_1,u) - f(t,x_2,u)\| = \left\| \begin{pmatrix} z_1 - z_2 \\ 0 \end{pmatrix} \right\|$   
Assumptions ODE hold: Theorem ODE applies!  $= |z_1 - z_2| \le \|x_1 - x_2\|$ 

Reusable rocket dynamics

$$egin{aligned} &rac{d}{dt}inom{r}{v}{m}(t)=inom{v}{rac{T_{ ext{max}}u}{m}-g_r}{-q\|u\|}inom{t}{u}(t) & ext{with controls} \quad u(\cdot)\in L^\infty([0,t_f],U), \ &U riangleq \{u\in\mathbb{R}^3: \ 0< u_{ ext{min}}\leq\|u\|\leq u_{ ext{max}}\}\ &(r,v,m)(0)=(r_0,v_0,m_0) \end{aligned}$$

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<u>It's your turn</u>: By restricting the analysis in the subset  $\{(r, v, m) : m \ge m_{\min} > 0\} \subseteq \mathbb{R}^3$ , show existence and uniqueness of solutions to this ODE (5-10 minutes)

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Solution: See the blackboard for details.

$$egin{aligned} &\|f(t,(0,0,m_{\min}),u)\|\leq \|g_r\|+\left(q+rac{T_{\max}}{m_{\min}}
ight)\|u\| & ext{and} \ &\|f(t,x_1,u)-f(t,x_2,u)\|\leq \max\left(1,rac{T_{\max}\|u\|}{m_{\min}^2}
ight)\|x_1-x_2\| & igcap_{52} \end{aligned}$$

# Today's detailed schedule

- 1. Revisit the concept of  $L^{\infty}([0, t_f], U)$ .
- 2. Revisit Ordinary Differential Equations (ODE): existence and uniqueness of solutions given non smooth controls.
- 3. Conditions for the existence of solutions to OCP and application to the reusable rocket landing problem.

 $\sum_{u(\cdot)\in L^\infty} h(x(t_f)) + \int_0^{t_f} f^0(t,x(t),u(t)) \; dt$ 

(Control) constraints $u(t)\in U\subseteq \mathbb{R}^m, \ \ t\in [0,t_f]$ 

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Dynamics + initial/final conditions

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1. From now on, we assume for every fixed control  $u(\cdot) \in L^{\infty}([0, t_f], U)$ ODE has a unique sol.  $x_u : [0, t_f] \to \mathbb{R}^n$ 

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- 2. OCP is called feasible if there exists at least one  $u(\cdot) \in L^{\infty}([0, t_f], U)$ such that the corresponding curve  $x_u : [0, t_f] \to \mathbb{R}^n$  satisfies  $g(x_u(t_f)) = 0$

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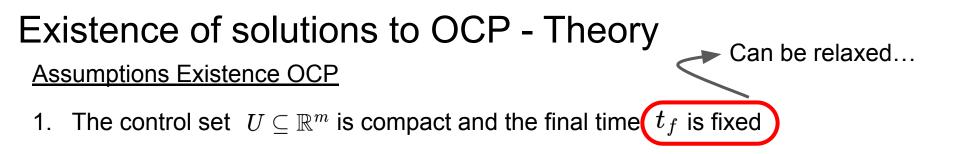
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Assumptions Existence OCP

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Can be relaxed...

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- 3. The epigraphs of extended velocities are <u>convex</u> for every  $(t,x) \in [0,t_f] imes \mathbb{R}^n$ :

$$V(t,x) riangleq \left\{ egin{pmatrix} f(t,x,u)\ f^0(t,x,u)+\gamma \end{pmatrix}: \ u\in U, \ \gamma\geq 0 
ight\} \subseteq \mathbb{R}^{n+1}$$

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Theorem Existence OCP (Existence of solutions to OCP).

Under the previous <u>Assumptions Existence OCP</u>, a feasible OCP has <u>at least one</u> optimal solution  $u^*(\cdot) \in L^{\infty}([0, t_f], U)$ , with optimal trajectory  $x_{u^*} : [0, t_f] \to \mathbb{R}^n$ .

Can be relaxed...

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} \;\; \int_0^{t_f} u(t)^2 \; dt, \;\; t_f>0 \; ext{fixed}$$

$$egin{cases} \dot{y}(t) = z(t) \ \dot{z}(t) = u(t) \end{cases}$$
 with  $egin{array}{c} y(0) = 1, \ z(0) = 0 \ y(t_f) = z(t_f) = 0 \end{cases}$ 

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"Reach the origin with zero final velocity and minimal energy"

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By assuming <u>OCP is feasible</u>, let us check <u>Assumptions Existence OCP</u> hold:

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By assuming <u>OCP is feasible</u>, let us check <u>Assumptions Existence OCP</u> hold:

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By assuming <u>OCP is feasible</u>, let us check <u>Assumptions Existence OCP</u> hold:

- 1. The control set U = [-1, 1] is compact (and the final time is clearly fixed)
- 2. We already checked <u>Assumptions ODE</u> hold. In addition, by denoting  $x \triangleq (y, z)$ the mappings h(x) = 0,  $f^0(t, x, u) = u^2$ , f(t, x, u) = (z, u), and g(x) = x are  $C^1$

3. It remains to show the convexity of the following sets, for every  $(t, x) \in [0, t_f] \times \mathbb{R}^2$ :

$$V(t,x)=\left\{egin{pmatrix} f(t,x,u)\ f^0(t,x,u)+\gamma\end{pmatrix}=egin{pmatrix} z\ u\ u\ u^2+\gamma\end{pmatrix}:\ u\in[-1,1],\ \gamma\geq 0
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$$V(t,x) = egin{cases} f(t,x,u)\ f^0(t,x,u) + \gamma \end{pmatrix} = egin{pmatrix} z\ u\ u\ u^2 + \gamma \end{pmatrix}: \ u\in [-1,1], \ \gamma\geq 0 \ \end{pmatrix} \subseteq \mathbb{R}^3 \ rac{ extsf{Proof:}}{ extsf{Proof:}} & extsf{Fix} \ (t,x)\in [0,t_f] imes \mathbb{R}^2 extsf{, and let} \ \lambda\in (0,1), \ egin{pmatrix} z\ u_1\ u_1^2+\gamma_1 \end{pmatrix}, \ egin{pmatrix} z\ u_2\ u_2^2+\gamma_2 \end{pmatrix}\in V(t,x) \end{cases}$$

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Then  $u_3 riangleq \lambda u_1 + (1-\lambda)u_2 \in [-1,1]$ 

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is convex, we have that  $\gamma_3 riangleq \lambda \gamma_1 + (1-\lambda)\gamma_2 + (\lambda u_1^2 + (1-\lambda)u_2^2 - u_3^2) \geq 0$ 

3. It remains to show the convexity of the following sets, for every  $(t, x) \in [0, t_f] \times \mathbb{R}^2$ :

$$\begin{split} V(t,x) &= \left\{ \begin{pmatrix} f(t,x,u) \\ f^0(t,x,u) + \gamma \end{pmatrix} = \begin{pmatrix} z \\ u \\ u^2 + \gamma \end{pmatrix} : \ u \in [-1,1], \ \gamma \ge 0 \right\} \subseteq \mathbb{R}^3 \\ \hline \text{Proof:} & \text{Fix } (t,x) \in [0,t_f] \times \mathbb{R}^2, \text{ and let } \lambda \in (0,1), \ \begin{pmatrix} z \\ u_1 \\ u_1^2 + \gamma_1 \end{pmatrix}, \ \begin{pmatrix} z \\ u_2 \\ u_2^2 + \gamma_2 \end{pmatrix} \in V(t,x) \\ \hline \text{Then } u_3 &\triangleq \lambda u_1 + (1-\lambda)u_2 \in [-1,1], \text{ and since the function } u \mapsto u^2 \\ \text{ is convex, we have that } \gamma_3 &\triangleq \lambda \gamma_1 + (1-\lambda)\gamma_2 + (\lambda u_1^2 + (1-\lambda)u_2^2 - u_3^2) \ge 0 \\ \hline \text{Thus } \lambda \begin{pmatrix} z \\ u_1 \\ u_1^2 + \gamma_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} z \\ u_2 \\ u_2^2 + \gamma_2 \end{pmatrix} = \begin{pmatrix} z \\ u_3 \\ u_3^2 + \gamma_3 \end{pmatrix} \in V(t,x) \end{split}$$

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$$\xrightarrow{\text{Then } u_3 \triangleq \lambda u_1 + (1-\lambda)u_2 \in [-1,1], \text{ and since the function } u \mapsto u^2$$

$$\text{ is convex, we have that } \gamma_3 \triangleq \lambda \gamma_1 + (1-\lambda)\gamma_2 + (\lambda u_1^2 + (1-\lambda)u_2^2 - u_3^2) \ge 0$$

$$\xrightarrow{\text{Thus}} \lambda \begin{pmatrix} z \\ u_1 \\ u_1^2 + \gamma_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} z \\ u_2 \\ u_2^2 + \gamma_2 \end{pmatrix} = \begin{pmatrix} z \\ u_3 \\ u_3^2 + \gamma_3 \end{pmatrix} \in V(t,x)$$

<u>Theorem Existence OCP</u> applies: if feasible, this OCP has at least one solution! 75

Original problem:

Minimal fuel use reusable rocket landing

 $\min_u \ -m(t_f)$ 

$$0 < u_{\min} \leq \|u(t)\| \leq u_{\max}$$

$$rac{d}{dt}egin{pmatrix} r \ v \ m \end{pmatrix}(t) = egin{pmatrix} rac{T_{ ext{max}u}}{m} - g_r \ -q\|u\| \end{pmatrix}(t)$$

 $egin{aligned} (r,v,m)(0) &= (r_0,v_0,m_0), \ &(r,v)(t_f) &= (0,0) \end{aligned}$ 

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New problem:

Minimal energy reusable rocket landing

Original problem:

Minimal fuel use reusable rocket landing

New problem:

Minimal energy reusable rocket landing

$$\min_u - m(t_f)$$

$$\min_{u\in L^\infty([0,t_f],U)} \hspace{0.1in} \int_0^{t_f} \|u(t)\|^2 \hspace{0.1in} dt, \hspace{0.1in} t_f>0 \hspace{0.1in} ext{fixed}$$

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$$U riangleq \{u\in \mathbb{R}^3: \; \|u\|\leq u_{ ext{max}}\}\subseteq \mathbb{R}^3$$

 $\min_{u\in L^\infty([0,t_f],U)} \;\; \int_0^{t_f} \|u(t)\|^2 \; dt, \quad t_f>0 ext{ fixed}$ 

Original problem:

Minimal fuel use reusable rocket landing

 $\min_{u} - m(t_f)$ 

 $0 < u_{\min} \leq \|u(t)\| \leq u_{\max}$ 

$$rac{d}{dt}egin{pmatrix} r \ v \ m \end{pmatrix}(t) = egin{pmatrix} rac{T_{ ext{max}u}}{m} - g_r \ -q \|u\| \end{pmatrix}(t)$$

 $egin{aligned} (r,v,m)(0) &= (r_0,v_0,m_0), \ &(r,v)(t_f) &= (0,0) \end{aligned}$ 

New problem:

Minimal energy reusable rocket landing

$$egin{aligned} &\min_{u\in L^\infty([0,t_f],U)} &\int_0^{t_f} \|u(t)\|^2 \, dt, \quad t_f > 0 ext{ fixed} \ &U &\triangleq \{u\in \mathbb{R}^3: \; \|u\| \leq u_{ ext{max}}\} \subseteq \mathbb{R}^3 \ &rac{d}{dt} inom{r}{v}(t) = inom{v(t)}{rac{T_{ ext{max}}u(t)}{ar{m}} - g_r}inom{} \ & inom{} \ & inom{r}{(r,v)(0)} = (r_0,v_0) \ & inom{r}{(r,v)(t_f)} = (0,0) \ & {}_{80} \end{aligned}$$

Original problem:

Minimal fuel use reusable rocket landing

New problem:

Minimal energy reusable rocket landing

$$egin{aligned} &\min_{u} & -m(t_{f}) \ &0 < u_{\min} \leq \|u(t)\| \leq u_{\max} \ &rac{d}{dt} \begin{pmatrix} r \ v \ m \end{pmatrix} (t) = egin{pmatrix} v \ rac{T_{\max}u}{m} - g_{r} \ -q\|u\| \end{pmatrix} (t) \ &(r,v,m)(0) = (r_{0},v_{0},m_{0}), \ &(r,v)(t_{f}) = (0,0) \end{aligned}$$

$$egin{aligned} &\min_{u\in L^\infty([0,t_f],U)} &\int_0^{t_f} \|u(t)\|^2 \,dt, \quad t_f>0 ext{ fixed} \ &U &\triangleq \{u\in \mathbb{R}^3: \; \|u\|\leq u_{\max}\}\subseteq \mathbb{R}^3 \ &rac{d}{dt}inom{r}{v}(t) = inom{v(t)}{rac{T_{\max}u(t)}{inom{m}}} - g_rigg) \ &iggl\{(r,v)(0) = (r_0,v_0) & iggledow ext{ mass!} \ &(r,v)(t_f) = (0,0) \end{aligned}$$

Minimal energy reusable rocket landing

 $\min_{u\in L^\infty([0,t_f],U)} \;\; \int_0^{t_f} \|u(t)\|^2 \; dt, \;\;\; t_f>0 \; ext{fixed}$ 

$$U riangleq \{u \in \mathbb{R}^3: \ \|u\| \leq u_{ ext{max}}\} \subseteq \mathbb{R}^3 \qquad rac{d}{dt} inom{r}{v}(t) = inom{v(t)}{rac{T_{ ext{max}}u(t)}{ar{m}} - g_r}inom{\{r,v)(0) = (r_0,v_0)}{(r,v)(t_f) = (0,0)}$$

<u>It's your turn</u>: By assuming that <u>this OCP is feasible</u>, show <u>Assumptions Existence OCP</u> hold true, so that OCP has at least one optimal solution (10/15 minutes).

Minimal energy reusable rocket landing

 $\min_{u \in L^\infty([0,t_f],U)} \;\; \int_0^{t_f} \|u(t)\|^2 \; dt, \;\;\; t_f > 0 \; ext{fixed}$ 

$$U riangleq \{u \in \mathbb{R}^3: \ \|u\| \leq u_{ ext{max}}\} \subseteq \mathbb{R}^3 \qquad rac{d}{dt} inom{r}{v}(t) = inom{v(t)}{ar{m}} - g_rigg
angle \qquad inom{(r,v)(0) = (r_0,v_0)}{ar{m}} + (r,v)(t_f) = (0,0)$$

<u>It's your turn</u>: By assuming that <u>this OCP is feasible</u>, show <u>Assumptions Existence OCP</u> hold true, so that OCP has at least one optimal solution (10/15 minutes).

Solution: See the blackboard for details.

1. The control set is clearly compact and the final time is clearly fixed

Minimal energy reusable rocket landing

 $\min_{u\in L^\infty([0,t_f],U)} \;\; \int_0^{t_f} \|u(t)\|^2 \; dt, \;\;\; t_f>0 \; ext{fixed}$ 

$$U riangleq \{u \in \mathbb{R}^3: \ \|u\| \leq u_{ ext{max}}\} \subseteq \mathbb{R}^3 \qquad rac{d}{dt} inom{r}{v}(t) = inom{v(t)}{ar{m}} - g_rigg
angle \qquad inom{(r,v)(0)}{\{r,v)(t_f) = (0,0)\}}$$

<u>It's your turn</u>: By assuming that <u>this OCP is feasible</u>, show <u>Assumptions Existence OCP</u> hold true, so that OCP has at least one optimal solution (10/15 minutes).

Solution: See the blackboard for details.

- 1. The control set is clearly compact and the final time is clearly fixed
- 2. We already checked <u>Assumptions ODE</u> hold. Also, by denoting  $x \triangleq (r, v) \in \mathbb{R}^6$

the mappings  $h(x) = 0, \; f^0(t,x,u) = \|u\|^2, \; f(t,x,u) = \left(rac{v(t)}{rac{T_{\max}u(t)}{ar{m}}} - g_r
ight), \; g(x) = x \; ext{are} \; \; C^1$ 

Minimal energy reusable rocket landing

 $\min_{u\in L^\infty([0,t_f],U)} \hspace{0.1in} \int_0^{t_f} \|u(t)\|^2 \hspace{0.1in} dt, \hspace{0.1in} t_f>0 \hspace{0.1in} ext{fixed}$ 

$$U riangleq \{u \in \mathbb{R}^3: \; \|u\| \leq u_{ ext{max}}\} \subseteq \mathbb{R}^3 \qquad rac{d}{dt} inom{r}{v}(t) = igg(rac{v(t)}{ar{m}} - g_rigg) \qquad igg\{(r,v)(0) = (r_0,v_0) \ (r,v)(t_f) = (0,0) \end{cases}$$

<u>It's your turn</u>: By assuming that <u>this OCP is feasible</u>, show <u>Assumptions Existence OCP</u> hold true, so that OCP has at least one optimal solution (10/15 minutes).

Solution: See the blackboard for details.

- 1. The control set is clearly compact and the final time is clearly fixed
- 2. We already checked <u>Assumptions ODE</u> hold. Also, by denoting  $x \triangleq (r, v) \in \mathbb{R}^6$

the mappings  $h(x) = 0, \; f^0(t,x,u) = \|u\|^2, \; f(t,x,u) = \begin{pmatrix} v(t) \\ \frac{T_{\max}u(t)}{\bar{m}} - g_r \end{pmatrix}, \; g(x) = x \; ext{are} \; \; C^1$ 

3. Replicate the argument in the previous example.

## Course schedule

- 1. Optimal Control Problems (OCP): review of ordinary differential equations; existence of solutions to OCP.
- 2. Optimality Conditions for OCP: the Maximum Principle and structure of optimal controls; application to reusable rocket landing (next class).
- 3. Python Session 1: real-world reusable rocket landing.
- 4. Python Session 2: training of neural networks through NeuralODE (this application might change depending on the course first outcomes).
- 5. Final presentation of the results.

#### End of lecture 1

Questions? Otherwise, see you in two weeks!