Advanced optimal control: from reusable rocket landing to efficient training of neural networks

IPSA course 2025

Riccardo Bonalli - riccardo.bonalli@cnrs.fr

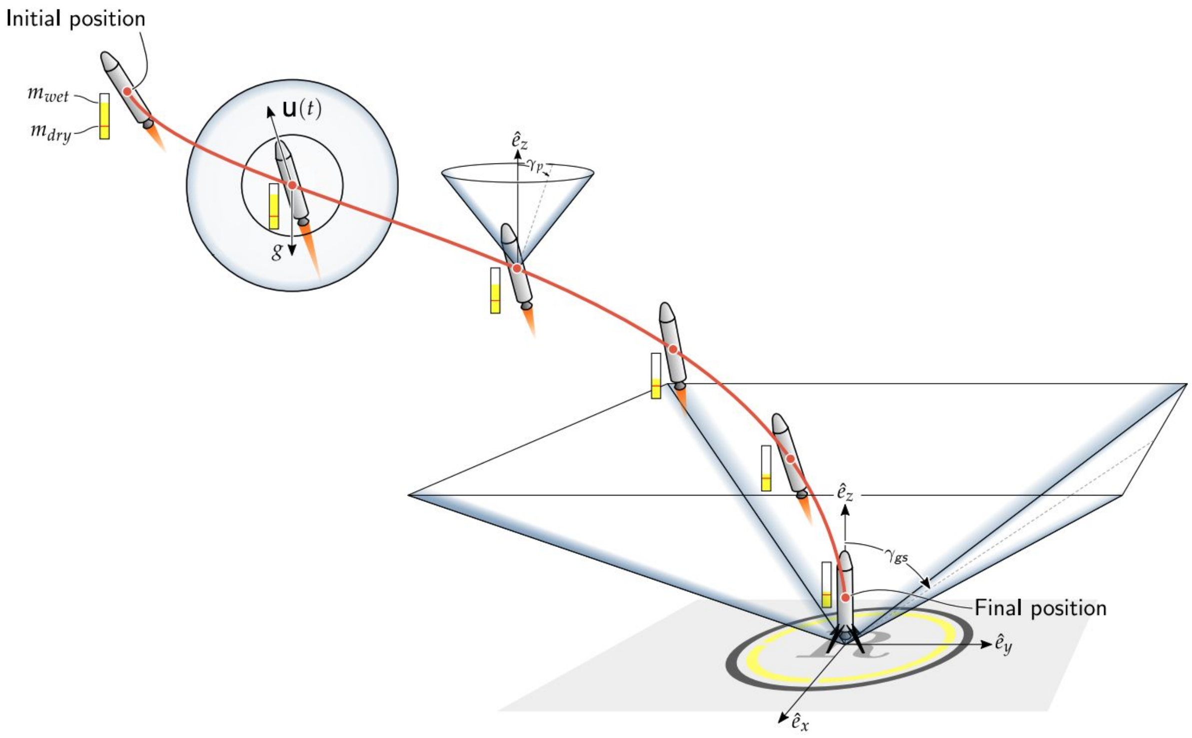
Laboratoire des Signaux et Systèmes
CNRS and Université Paris-Saclay

L2S Laboratoire Signaux & Systèmes

cnrs

université PARIS-SACLAY

Example: reusable rocket



Example: reusable rocket

r = position

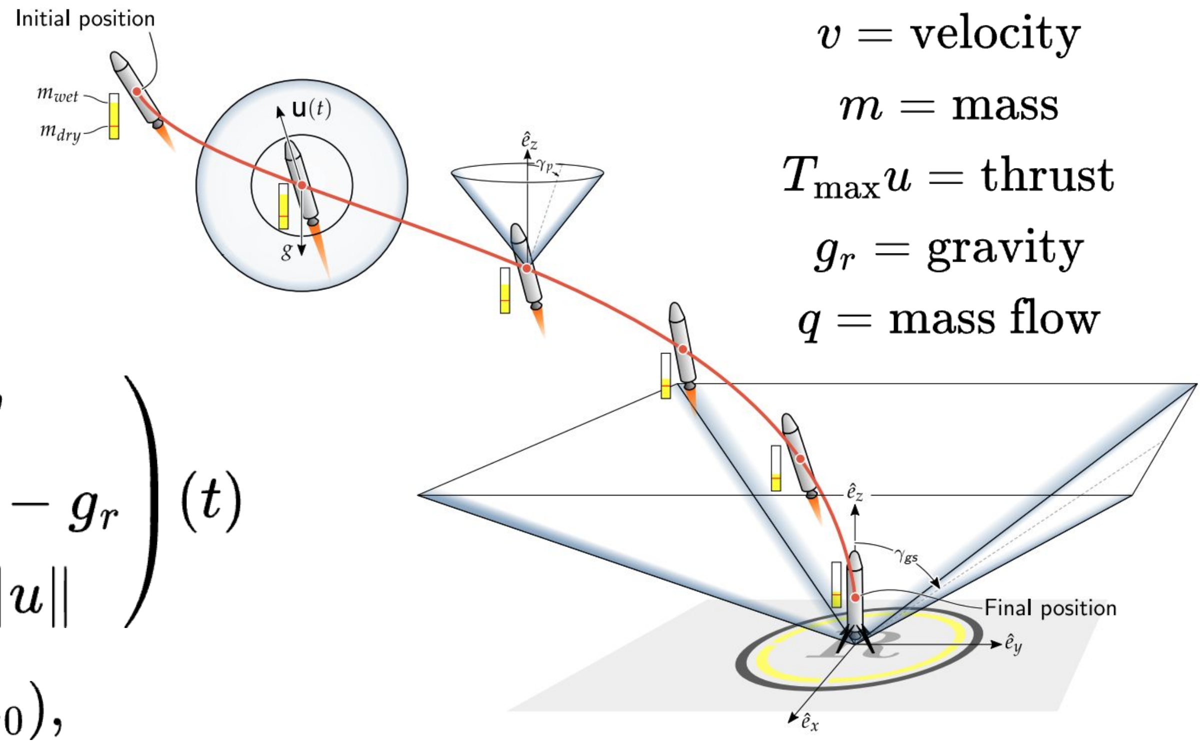
v = velocity

m = mass

$T_{\max} u$ = thrust

g_r = gravity

q = mass flow



$$\frac{d}{dt}\begin{pmatrix} r \\ v \\ m \end{pmatrix}(t) = \begin{pmatrix} v \\ \frac{T_{\max} u}{m} - g_r \\ -q\|u\| \end{pmatrix}(t)$$

$$(r, v, m)(0) = (r_0, v_0, m_0),$$

$$(r, v)(t_f) = (0, 0)$$

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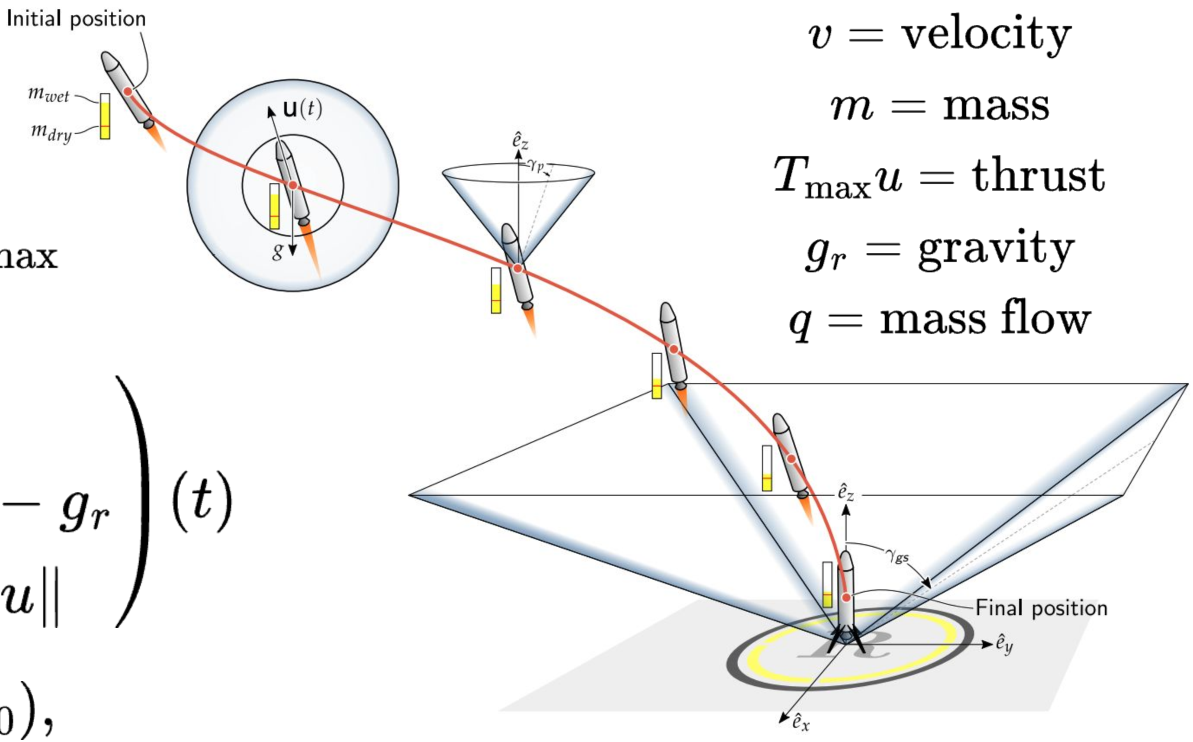

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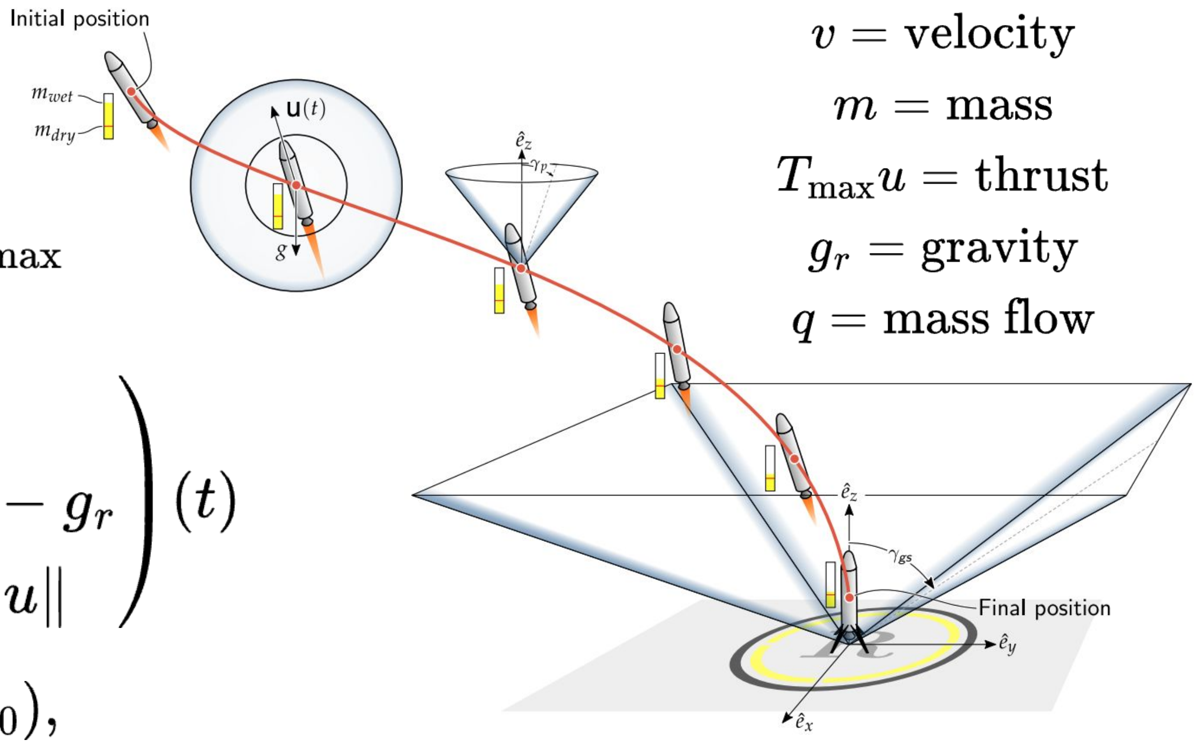

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Dynamics + initial/final conditions

$$\begin{cases} x(t) \in \mathbb{R}^n, \quad t \in [0, t_f], \\ \dot{x}(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \quad g(x(t_f)) = 0 \end{cases}$$

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Example: reusable rocket

Cost (functional)

$$\min_{u} \ -m(t_f)$$

$$\min_{u(\cdot)\in L^\infty} \ h(x(t_f)) + \int_0^{t_f} f^0(t, x(t), u(t))\, dt$$

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Optimal Control Problem (OCP)

Will learn theory and algorithms
to efficiently solve OCP

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Key difficulties

1. OCP is infinite dimensional
   (controls are in L infinity)

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1. OCP is infinite dimensional
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2. OCP is non-convex
   (dynamics and cost are non-convex)

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Ultimate goal

Get a comprehensive understanding of the key challenges faced by contemporary aerospace control engineers, addressing both theoretical and practical aspects

Course schedule

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5. Final presentation of the results (the SOLE evaluation step).
   → Group evaluation: please form groups of 2 / 3 people and fill the Excel file on Moodle ASAP!

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Lecture 1 - Existence of solutions to OCP

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$$h : \mathbb{R}^n \to \mathbb{R}$$

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Key facts

1. The final time t_f may be free
2. All mappings are continuous…

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Key facts

1. The final time t_f may be free
2. All mappings are continuous… except for controls!
3. $u : [0, t_f] \to U$ such that $u(\cdot) \in L^\infty([0, t_f], U)$

Today's detailed schedule

1. Revisit the concept of $L^\infty([0, t_f], U)$.

2. Revisit Ordinary Differential Equations (ODE): existence and uniqueness of solutions given non smooth controls.

3. Conditions for the existence of solutions to OCP and application to the reusable rocket landing problem.

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Quick reminders on Lp spaces - Theory

In this course we will often deal with:

$$L^\infty([0,t_f],U) \triangleq \{u:[0,t_f]\to U\subseteq\mathbb{R}^m \text{ s.t.}$$
$$u(\cdot) \text{ is } \textit{measurable} \quad \text{and} \quad \|u(\cdot)\|_{L^\infty}<\infty\}$$

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1. "Basically", integrals like $\int_0^t \varphi(s,u(s))\,ds$ are well-defined for every $t \in [0,t_f]$ and $\varphi \in C^0([0,t_f]\times\mathbb{R}^m,\mathbb{R}^\ell)$

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2. The "supremum" norm $\|u(\cdot)\| \triangleq \sup_{t\in[0,t_f]} \|u(t)\|$ is well-defined and finite.

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Example: bang-bang controls

$$m = 1, \quad U \triangleq [-1,1] \subseteq \mathbb{R}, \quad t_{sw} \in [0,t_f]$$

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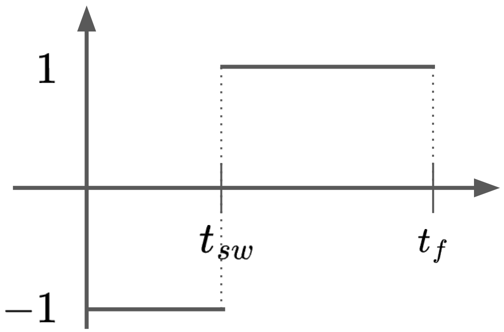
$$u_{t_{sw}}(t) \triangleq \begin{cases} -1, & t \leq t_{sw} \\ 1, & t > t_{sw} \end{cases}, \quad t \in [0,t_f]$$



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(note that $x(\cdot)$ depends on both $u(\cdot)$ and x_0 !)

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2. We need to answer the following question:

Under what assumptions on f does a solution to ODE exist uniquely?

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$B. \quad \|f(t,x,u) - f(t,y,u)\| \leq L(\|u\|)\|x-y\|, \quad (t,u) \in [0,t_f] \times U, \ x,y \in \mathbb{R}^n \quad (Lipschitzianity)$

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ODE $\begin{cases} \dot{x}(t) = f(t,x(t),u(t)), \\ t \in [0,t_f], \quad x(0) = x_0 \end{cases}$ where $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is "at least" continuous

3. <u>Assumptions ODE</u> - There exist $\bar{x} \in \mathbb{R}^n$ and a C^0 function $L(\cdot) > 0$ such that:

$A. \quad \|f(t,\bar{x},u)\| \le L(\|u\|), \quad (t,u) \in [0,t_f] \times U \quad (u - boundness)$

$B. \quad \|f(t,x,u) - f(t,y,u)\| \le L(\|u\|)\|x-y\|, \quad (t,u) \in [0,t_f] \times U, \ x,y \in \mathbb{R}^n \quad (Lipschitzianity)$

Could be replaced with open subset $A \subseteq \mathbb{R}^n$

Reminders on ODE - Theory

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4. Theorem ODE (Existence and uniqueness of solutions to ODE).

Under Assumptions ODE, ODE has a unique solution $x : [0,t_f] \to \mathbb{R}^n$.

Reminders on ODE - Examples

Double integrator - Used in electronic circuit modeling

$$\begin{cases} \dot{y}(t) = z(t) \\ \dot{z}(t) = u_{t_{sw}}(t) \\ t \in [0, t_f],\ x(0) = y(0) = 0 \end{cases}$$

where $$u_{t_{sw}}(t) \triangleq \begin{cases} -1, & t \leq t_{sw} \\ 1, & t > t_{sw} \end{cases}, \quad t \in [0, t_f]$$

(clearly $u_{t_{sw}} \in L^{\infty}([0, t_f], [-1, 1])$)

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$$n = 2, m = 1, \ \ x \triangleq (y, z), \ \ f(t, x, u) \triangleq \begin{pmatrix} z \\ u \end{pmatrix}$$

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$$\|f(t, 0, u)\| = |u| \quad \text{and} \quad \|f(t, x_1, u) - f(t, x_2, u)\| = \left\| \begin{pmatrix} z_1 - z_2 \\ 0 \end{pmatrix} \right\|$$

$$= |z_1 - z_2| \leq \|x_1 - x_2\|$$

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Assumptions ODE hold: Theorem ODE applies!

Reminders on ODE - Examples

Reusable rocket dynamics

$$\frac{d}{dt}\begin{pmatrix} r \\ v \\ m \end{pmatrix}(t) = \begin{pmatrix} v \\ \frac{T_{\max} u}{m} - g_r \\ -q\|u\| \end{pmatrix}(t)$$ with controls $u(\cdot) \in L^\infty([0, t_f], U)$,

$$U \triangleq \{u \in \mathbb{R}^3 :\ 0 < u_{\min} \leq \|u\| \leq u_{\max}\}$$

$$(r, v, m)(0) = (r_0, v_0, m_0)$$

Reminders on ODE - Examples

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<u>It's your turn</u>: By restricting the analysis in the subset $\{(r, v, m) : \ m \geq m_{\min} > 0\} \subseteq \mathbb{R}^3$, show existence and uniqueness of solutions to this ODE (5-10 minutes)

Reminders on ODE - Examples

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It’s your turn: By restricting the analysis in the subset $\{(r, v, m) : \ m \geq m_{\min} > 0\} \subseteq \mathbb{R}^3$, show existence and uniqueness of solutions to this ODE (5-10 minutes)

Solution: See the blackboard for details.

$$\|f(t, (0, 0, m_{\min}), u)\| \leq \|g_r\| + \left(q + \frac{T_{\max}}{m_{\min}}\right)\|u\| \quad \text{and}$$

$$\|f(t, x_1, u) - f(t, x_2, u)\| \leq \max\left(1, \frac{T_{\max}\|u\|}{m_{\min}^2}\right)\|x_1 - x_2\| \longrightarrow \text{Theorem ODE applies!}$$

Today’s detailed schedule

1. Revisit the concept of $L^\infty([0, t_f], U)$.

2. Revisit Ordinary Differential Equations (ODE): existence and uniqueness of solutions given non smooth controls.

3. Conditions for the existence of solutions to OCP and application to the reusable rocket landing problem.

Existence of solutions to OCP - Theory

Cost (functional)

$$\min_{u(\cdot)\in L^\infty} \; h(x(t_f)) + \int_0^{t_f} f^0(t, x(t), u(t))\, dt$$

(Control) constraints

$$u(t) \in U \subseteq \mathbb{R}^m, \quad t \in [0, t_f]$$

Dynamics + initial/final conditions

$$\begin{cases} x(t) \in \mathbb{R}^n, \quad t \in [0, t_f], \\ \dot{x}(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \quad g(x(t_f)) = 0 \end{cases}$$

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1. From now on, we assume for every fixed control $u(\cdot) \in L^\infty([0, t_f], U)$ ODE has a unique sol. $x_u : [0, t_f] \to \mathbb{R}^n$

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2. OCP is called feasible if there exists at least one $u(\cdot) \in L^\infty([0, t_f], U)$ such that the corresponding curve $x_u : [0, t_f] \to \mathbb{R}^n$ satisfies $g(x_u(t_f)) = 0$

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3. Key question: Under what conditions a feasible OCP has at least one optimal (control) solution $u^*(\cdot) \in L^\infty([0, t_f], U)$?

Existence of solutions to OCP - Theory

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Maybe more than one!!!

Existence of solutions to OCP - Theory

Assumptions Existence OCP

Existence of solutions to OCP - Theory

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1. The control set $U \subseteq \mathbb{R}^m$ is compact and the final time t_f is fixed

Existence of solutions to OCP - Theory

Can be relaxed…

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1. The control set $U \subseteq \mathbb{R}^m$ is compact and the final time t_f is fixed
2. All the mappings $h, f^0, f, \text{ and } g$ are C^1, and Assumptions ODE hold

Existence of solutions to OCP - Theory

Can be relaxed…

Assumptions Existence OCP

1. The control set $U \subseteq \mathbb{R}^m$ is compact and the final time t_f is fixed
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3. The epigraphs of extended velocities are convex for every $(t,x) \in [0, t_f] \times \mathbb{R}^n$:

$$V(t,x) \triangleq \left\{ \begin{pmatrix} f(t,x,u) \\ f^0(t,x,u) + \gamma \end{pmatrix} : \ u \in U, \ \gamma \geq 0 \right\} \subseteq \mathbb{R}^{n+1}$$

Existence of solutions to OCP - Theory

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Theorem Existence OCP (Existence of solutions to OCP).

Under the previous Assumptions Existence OCP, a feasible OCP has at least one optimal solution $u^*(\cdot) \in L^\infty([0,t_f], U)$, with optimal trajectory $x_{u^*} : [0, t_f] \to \mathbb{R}^n$.

Existence of solutions to OCP - Examples

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} \int_0^{t_f} u(t)^2 \, dt, \quad t_f > 0 \text{ fixed}$$

$$\begin{cases} \dot{y}(t) = z(t) \\ \dot{z}(t) = u(t) \end{cases} \quad \text{with} \quad \begin{cases} y(0) = 1, \;\; z(0) = 0 \\ y(t_f) = z(t_f) = 0 \end{cases}$$

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"Reach the origin with zero final velocity and minimal energy"

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By assuming OCP is feasible, let us check Assumptions Existence OCP hold:

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By assuming OCP is feasible, let us check Assumptions Existence OCP hold:

1. The control set $U = [-1, 1]$ is compact (and the final time is clearly fixed)

2. We already checked Assumptions ODE hold. In addition, by denoting $x \triangleq (y, z)$

the mappings $h(x) = 0$, $f^0(t, x, u) = u^2$, $f(t, x, u) = (z, u)$, and $g(x) = x$ are C^1

Existence of solutions to OCP - Examples

3. It remains to show the convexity of the following sets, for every $(t, x) \in [0, t_f] \times \mathbb{R}^2$:

$$V(t,x) = \left\{ \begin{pmatrix} f(t,x,u) \\ f^0(t,x,u) + \gamma \end{pmatrix} = \begin{pmatrix} z \\ u \\ u^2 + \gamma \end{pmatrix} \,:\, u \in [-1,1],\ \gamma \geq 0 \right\} \subseteq \mathbb{R}^3$$

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Proof: Fix $(t, x) \in [0, t_f] \times \mathbb{R}^2$, and let $\lambda \in (0, 1)$, $\begin{pmatrix} z \\ u_1 \\ u_1^2 + \gamma_1 \end{pmatrix}, \begin{pmatrix} z \\ u_2 \\ u_2^2 + \gamma_2 \end{pmatrix} \in V(t, x)$

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Then $u_3 \triangleq \lambda u_1 + (1-\lambda) u_2 \in [-1,1]$

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Then $u_3 \triangleq \lambda u_1 + (1-\lambda)u_2 \in [-1,1]$, and since the function $u \mapsto u^2$ is convex, we have that $\gamma_3 \triangleq \lambda\gamma_1 + (1-\lambda)\gamma_2 + (\lambda u_1^2 + (1-\lambda)u_2^2 - u_3^2) \geq 0$

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Then $u_3 \triangleq \lambda u_1 + (1-\lambda) u_2 \in [-1,1]$, and since the function $u \mapsto u^2$ is convex, we have that $\gamma_3 \triangleq \lambda \gamma_1 + (1-\lambda)\gamma_2 + (\lambda u_1^2 + (1-\lambda) u_2^2 - u_3^2) \geq 0$

Thus

$$\lambda \begin{pmatrix} z \\ u_1 \\ u_1^2 + \gamma_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} z \\ u_2 \\ u_2^2 + \gamma_2 \end{pmatrix} = \begin{pmatrix} z \\ u_3 \\ u_3^2 + \gamma_3 \end{pmatrix} \in V(t,x)$$

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$$V(t,x) = \left\{ \begin{pmatrix} f(t,x,u) \\ f^0(t,x,u) + \gamma \end{pmatrix} = \begin{pmatrix} z \\ u \\ u^2 + \gamma \end{pmatrix} \,:\, u \in [-1,1],\ \gamma \geq 0 \right\} \subseteq \mathbb{R}^3$$

Proof: Fix $(t,x) \in [0,t_f] \times \mathbb{R}^2$, and let $\lambda \in (0,1)$, $\begin{pmatrix} z \\ u_1 \\ u_1^2 + \gamma_1 \end{pmatrix}, \begin{pmatrix} z \\ u_2 \\ u_2^2 + \gamma_2 \end{pmatrix} \in V(t,x)$

Then $u_3 \triangleq \lambda u_1 + (1-\lambda) u_2 \in [-1,1]$, and since the function $u \mapsto u^2$ is convex, we have that $\gamma_3 \triangleq \lambda \gamma_1 + (1-\lambda)\gamma_2 + (\lambda u_1^2 + (1-\lambda) u_2^2 - u_3^2) \geq 0$

Thus

$$\lambda \begin{pmatrix} z \\ u_1 \\ u_1^2 + \gamma_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} z \\ u_2 \\ u_2^2 + \gamma_2 \end{pmatrix} = \begin{pmatrix} z \\ u_3 \\ u_3^2 + \gamma_3 \end{pmatrix} \in V(t,x)$$

Theorem Existence OCP applies: if feasible, this OCP has at least one solution!

Existence of solutions to OCP - Examples

Original problem:

Minimal fuel use reusable rocket landing

$$\min_u \; -m(t_f)$$

$$0 < u_{\min} \leq \|u(t)\| \leq u_{\max}$$

$$\frac{d}{dt}\begin{pmatrix} r \\ v \\ m \end{pmatrix}(t) = \begin{pmatrix} v \\ \frac{T_{\max} u}{m} - g_r \\ -q\|u\| \end{pmatrix}(t)$$

$$(r, v, m)(0) = (r_0, v_0, m_0),$$

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New problem:

Minimal energy reusable rocket landing

$$\min_{u \in L^\infty([0,t_f],U)} \int_0^{t_f} \|u(t)\|^2 \, dt, \quad t_f > 0 \text{ fixed}$$

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Some fixed mass!

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<u>It's your turn</u>: By assuming that <u>this OCP is feasible</u>, show <u>Assumptions Existence OCP</u> hold true, so that OCP has at least one optimal solution (10/15 minutes).

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Solution: See the blackboard for details.

1. The control set is clearly compact and the final time is clearly fixed

Existence of solutions to OCP - Examples

Minimal energy reusable rocket landing

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<u>Solution</u>: See the blackboard for details.

1. The control set is clearly compact and the final time is clearly fixed
2. We already checked <u>Assumptions ODE</u> hold. Also, by denoting $x \triangleq (r,v) \in \mathbb{R}^6$

the mappings $h(x) = 0, \ f^0(t,x,u) = \|u\|^2, \ f(t,x,u) = \begin{pmatrix} v(t) \\ \frac{T_{\max} u(t)}{\bar{m}} - g_r \end{pmatrix}, \ g(x) = x$ are C^1

Existence of solutions to OCP - Examples

Minimal energy reusable rocket landing

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3. Replicate the argument in the previous example.

Course schedule

1. Optimal Control Problems (OCP): review of ordinary differential equations; existence of solutions to OCP.

2. Optimality Conditions for OCP: the Maximum Principle and structure of optimal controls; application to reusable rocket landing (next class).

3. Python Session 1: real-world reusable rocket landing.

4. Python Session 2: training of neural networks through NeuralODE (this application might change depending on the course first outcomes).

5. Final presentation of the results.

End of lecture 1

Questions?
Otherwise, see you in two weeks!