Advanced optimal control: from reusable rocket landing to efficient training of neural networks

IPSA course 2025

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Laboratoire des Signaux et Systèmes
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Course schedule

1. Optimal Control Problems (OCP): review of ordinary differential equations; existence of solutions to OCP.

2. Optimality Conditions for OCP: the Maximum Principle and structure of optimal controls; application to reusable rocket landing.

3. Python Session 1: real-world reusable rocket landing.

4. Python Session 2: training of neural networks through NeuralODE (this application might change depending on the course first outcomes).

5. Final presentation of the results.

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Recap

Cost (functional)

$$\min_{u(\cdot)\in L^\infty} \; h(x(t_f)) + \int_0^{t_f} f^0(t, x(t), u(t)) \, dt$$

(Control) constraints

$$u(t) \in U \subseteq \mathbb{R}^m, \quad t \in [0, t_f]$$

Dynamics + initial/final conditions

$$\begin{cases} x(t) \in \mathbb{R}^n, \quad t \in [0, t_f], \\ \dot{x}(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \quad g(x(t_f)) = 0 \end{cases}$$

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1. We studied conditions under which ODE has a unique sol. $x_u : [0, t_f] \to \mathbb{R}^n$ for any fixed $u(\cdot) \in L^\infty([0, t_f], U)$.

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3. Today's topic: characterize optimal solutions $u^*(\cdot) \in L^\infty([0, t_f], U)$ to OCP via necessary conditions of optimality.

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Lecture 2 - Characterize the structure of solutions to OCP

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Today’s detailed schedule

1. Necessary conditions for optimality: the Pontryagin Maximum Principle (PMP).

2. Learn how to apply the PMP through simple optimal control problems.

3. Characterize optimal controls for the reusable rocket landing problem.

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Key insight: OCP is a (infinite-dimensional) constrained optimization problem.

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Let us introduce such conditions for optimality for OCP, and characterize optimal controls!

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New variables: <u>multipliers</u>!

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a. $\nabla_x H = \text{gradient w.r.t. } x$
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b. <u>Backward linear ODE</u>: unique solution exists

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3. <u>Theorem PMP</u> (necessary conditions for optimality).

   Let (u^*, x^*) be an optimal control-trajectory pair solution to OCP. There exists a pair $(p^0, \mathfrak{p}) \in \{-1, 0\} \times \mathbb{R}^\ell$ such that the following conditions hold true:

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   c. Maximality condition: $u^*(t) \in \arg\max_{u \in U} H(t, x^*(t), u, p^*(t), p^0)$

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   b. Non-triviality: $(p^*(t), p^0) \neq 0$ for all $t \in [0, t_f]$

   c. Maximality condition: $u^*(t) \in \arg\max_{u \in U} H(t, x^*(t), u, p^*(t), p^0)$

   d. Final condition (only if t_f is free!!!): $\max_{u \in U} H(t_f, x^*(t_f), u, p^*(t_f), p^0) = 0$

The PMP - Theory

3. <u>Theorem PMP</u> (necessary conditions for optimality).

   Let (u^*, x^*) be an optimal control-trajectory pair solution to OCP. There exists a pair $(p^0, \mathfrak{p}) \in \{-1, 0\} \times \mathbb{R}^\ell$ such that the following conditions hold true:

   a. Adjoint and adjoint system:

      Let $p^* : [0, t_f] \to \mathbb{R}^n$ be the unique solution the the <u>adjoint system</u>:

$$\begin{cases} \dot{p}^*(t) = -\nabla_x H(t, x^*(t), u^*(t), p^*(t), p^0), \\ p^*(t_f) = p^0 \nabla_x h(x^*(t_f)) + \mathfrak{p}^\top J_x g(x^*(t_f)) \end{cases}$$

   b. Non-triviality: $(p^*(t), p^0) \neq 0$ for all $t \in [0, t_f]$

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   d. Final condition (<u>only if t_f is free</u>!!!): $\max_{u \in U} H(t_f, x^*(t_f), u, p^*(t_f), p^0) = 0$

Used to find simpler forms of $u^*(\cdot)$.

Today's detailed schedule

1. Necessary conditions for optimality: the Pontryagin Maximum Principle (PMP).

2. Learn how to apply the PMP through simple optimal control problems.

3. Characterize optimal controls for the reusable rocket landing problem.

The PMP - Examples

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} \int_0^{t_f} u(t)^2\, dt, \quad t_f > 0 \text{ fixed}$$

$$\begin{cases} \dot{y}(t) = z(t) \\ \dot{z}(t) = u(t) \end{cases} \quad \text{with} \quad \begin{cases} y(0) = 1, \ \ z(0) = 0 \\ y(t_f) = z(t_f) = 0 \end{cases}$$

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Assuming that there exists an optimal control-trajectory pair (u^*, x^*), let us apply the PMP and find computationally simpler forms of optimal controls. There are 3 steps:

The PMP - Examples

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1. Write the Hamiltonian:

$$x \triangleq (y, z) \in \mathbb{R}^2, \quad u \in \mathbb{R}, \quad p \triangleq (p_y, p_z) \in \mathbb{R}^2, \quad p^0 \in \{0, -1\}$$

The PMP - Examples

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$$H(t, x, u, p, p^0) = p^\top f(t, x, u) + p^0 f^0(t, x, u) = p_y z + p_z u + p^0 u^2$$

The PMP - Examples

2. Given the Hamiltonian $H(t, x, u, p, p^0) = p_y z + p_z u + p^0 u^2$, there exist a vector $\mathfrak{p} \in \mathbb{R}^2$ and an adjoint $p^* : [0, t_f] \to \mathbb{R}^2$ satisfying the adjoint system:

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In particular, we compute

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$$\begin{pmatrix} \dot{p}_y^*(t) \\ \dot{p}_z^*(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial H}{\partial y}(t, x^*(t), u^*(t), p^*(t), p^0) \\ -\dfrac{\partial H}{\partial z}(t, x^*(t), u^*(t), p^*(t), p^0) \end{pmatrix} = \begin{pmatrix} 0 \\ -p_y^*(t) \end{pmatrix}$$

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Thus:
$$p_y^*(t) = \mathfrak{p}_y, \quad p_z^*(t) = \mathfrak{p}_z + \mathfrak{p}_y(t_f - t), \quad t \in [0, t_f]$$

The PMP - Examples

3. Given the Hamiltonian $H(t, x, u, p, p^0) = p_y z + p_z u + p^0 u^2$ and the adjoint $p_y^*(t) = \mathfrak{p}_y, \quad p_z^*(t) = \mathfrak{p}_z + \mathfrak{p}_y(t_f - t)$, solve the maximality condition

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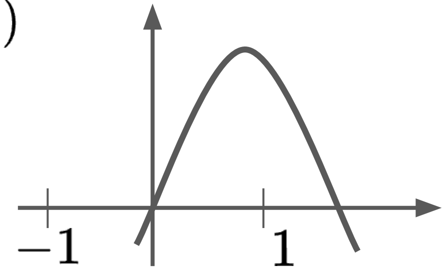

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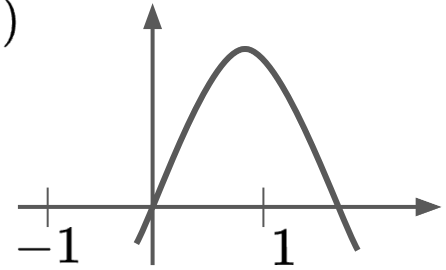
Solutions depend on the value of p^0 !



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a. Case $p^0 = -1$

Maximizing a parabola:

$$u^*(t) = \arg\max_{u \in [-1,1]} (p_z^*(t)u - u^2)$$

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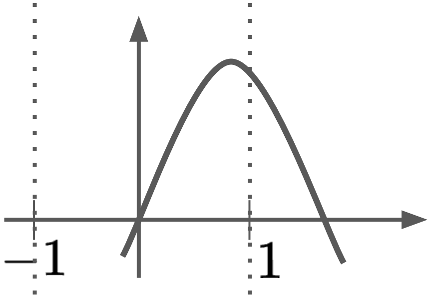

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Maximizing a parabola: maximum at the vertex if in [-1,1]

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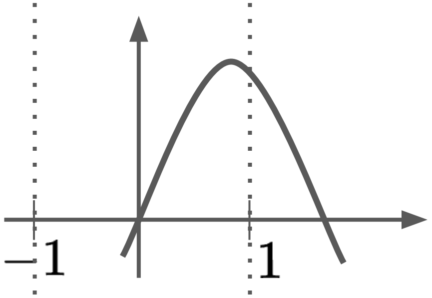

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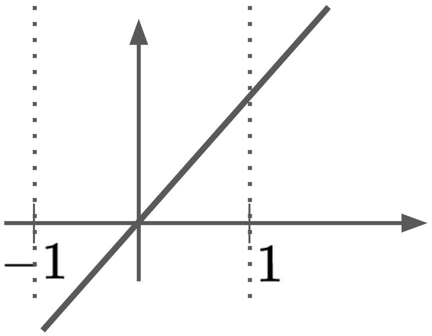
Maximizing a straight line in [-1,1]



$$u^*(t) = \arg\max_{u \in [-1,1]} p_z^*(t)u$$

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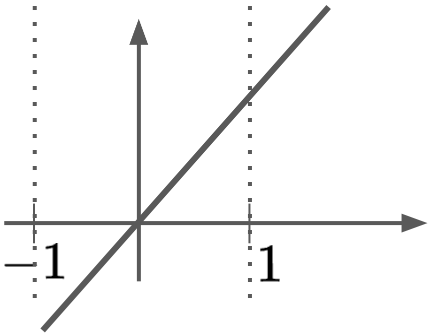

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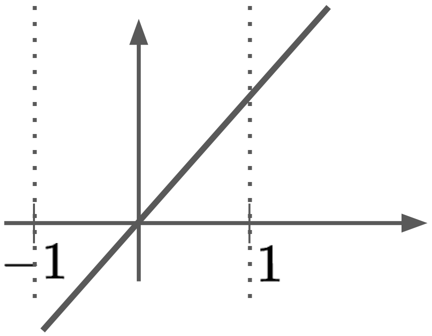

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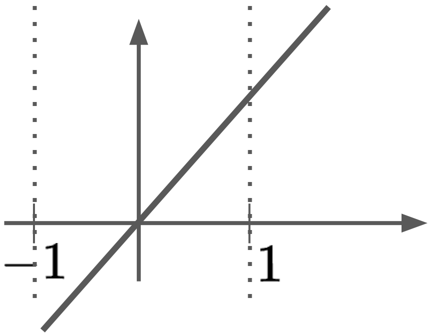

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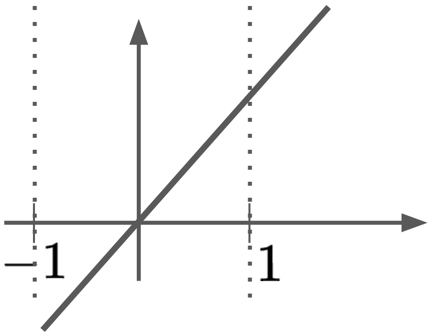

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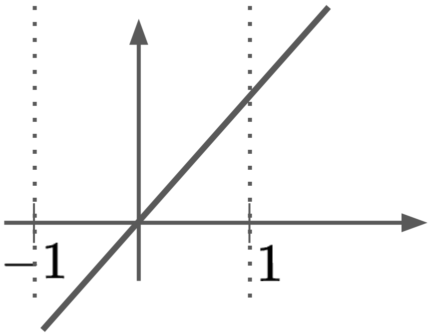

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Finally, we would obtain $p_y^*(t) = p_z^*(t) = p^0 = 0, \ t \in [0, t_f]$, contradiction!

The PMP - Examples

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} \int_0^{t_f} u(t)^2 \, dt, \quad t_f > 0 \text{ fixed}$$

$$\begin{cases} \dot{y}(t) = z(t) \\ \dot{z}(t) = u(t) \end{cases} \quad \text{with} \quad \begin{cases} y(0) = 1, \;\; z(0) = 0 \\ y(t_f) = z(t_f) = 0 \end{cases}$$

We have finally showed that optimal controls for this OCP can only take two forms:

The PMP - Examples

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} \int_0^{t_f} u(t)^2 \, dt, \quad t_f > 0 \text{ fixed}$$

$$\begin{cases} \dot{y}(t) = z(t) \\ \dot{z}(t) = u(t) \end{cases} \quad \text{with} \quad \begin{cases} y(0) = 1, \ \ z(0) = 0 \\ y(t_f) = z(t_f) = 0 \end{cases}$$

We have finally showed that optimal controls for this OCP can only take two forms:

By denoting $a \triangleq t_f - \dfrac{2 - \mathfrak{p}_z}{\mathfrak{p}_y}, \quad b \triangleq t_f + \dfrac{2 + \mathfrak{p}_z}{\mathfrak{p}_y}, \ \text{ and } \ c \triangleq t_f + \dfrac{\mathfrak{p}_z}{\mathfrak{p}_y}$, we have

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$$u^*(t) = \begin{cases} \dfrac{2}{a-b}t - \dfrac{a+b}{a-b}, \;\; \text{if } t \in [a,b] \\ -1, \;\; \text{if } t > b \\ 1, \;\; \text{if } t < a \end{cases}$$

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Just find $0 \leq a < b \leq t_f$ OR $0 \leq c \leq t_f$ numerically!!!

Existence of solutions to OCP - Examples

Minimal energy reusable rocket landing

$$\min_{u \in L^\infty([0,t_f],U)} \int_0^{t_f} \|u(t)\|^2 \, dt, \quad t_f > 0 \text{ fixed}$$

$$U \triangleq \{u \in \mathbb{R}^3 : \ \|u\| \leq u_{\max}\} \subseteq \mathbb{R}^3$$

$$\frac{d}{dt}\begin{pmatrix} r \\ v \end{pmatrix}(t) = \begin{pmatrix} v(t) \\ \dfrac{T_{\max} u(t)}{\bar{m}} - g_r \end{pmatrix} \qquad \begin{cases} (r,v)(0) = (r_0, v_0) \\ (r,v)(t_f) = (0,0) \end{cases}$$

It's your turn: By assuming that this OCP has at least one optimal control-trajectory pair (u^*, x^*), apply the PMP and show $u^*(\cdot)$ is of the form (20/30 minutes):

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$$u^*(t) = \begin{cases} \mathfrak{f}(t), & \text{if } \|\mathfrak{f}(t)\| \leq u_{\max} \\ u_{\max} \dfrac{\mathfrak{f}(t)}{\|\mathfrak{f}(t)\|}, & \text{if } \|\mathfrak{f}(t)\| > u_{\max} \end{cases}$$

OR

$$u^*(t) = u_{\max} \frac{\mathfrak{f}(t)}{\|\mathfrak{f}(t)\|}$$

Solution: See the blackboard.

Today’s detailed schedule

1. Necessary conditions for optimality: the Pontryagin Maximum Principle (PMP).
2. Learn how to apply the PMP through simple optimal control problems.
3. Characterize optimal controls for the reusable rocket landing problem.

Characterize optimal controls for rocket landing

Minimal fuel consumption reusable rocket landing (with fixed or free final time)

$$\min_u \; -m(t_f)$$

$$\frac{d}{dt}\begin{pmatrix} r \\ v \\ m \end{pmatrix}(t) = \begin{pmatrix} v \\ \frac{T_{\max} u}{m} - g_r \\ -q\|u\| \end{pmatrix}(t) \qquad (r, v, m)(0) = (r_0, v_0, m_0),$$

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1-2. For the Hamiltonian and the adjoint system we have (see the blackboard):

$$x \triangleq (r, v, m) \in \mathbb{R}^7, \quad u \in \mathbb{R}^3, \quad p \triangleq (p_r, p_v, p_m) \in \mathbb{R}^7, \quad p^0 \in \{0, -1\}$$

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$$\begin{cases} \dot{p}_r^*(t) = 0 \\ p_r^*(t_f) = \mathfrak{p}_r \end{cases} \qquad \begin{cases} \dot{p}_v^*(t) = -p_r^*(t) \\ p_v^*(t_f) = \mathfrak{p}_v \end{cases} \qquad \begin{cases} \dot{p}_m^*(t) = \dfrac{p_v^*(t) \cdot u^*(t)}{m(t)^2} \\ p_m^*(t_f) = -p^0 \end{cases}$$

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$u = \alpha\, n,\ \ \alpha = \|u\|,\ \ n = \frac{u}{\|u\|}$

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(if not, similar results but complex study)

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$$\Psi(t) \triangleq \frac{\|p_v^*(t)\|}{m(t)} - p_m^*(t)q$$

What if $\Psi(t) = 0$???

Characterize optimal controls for rocket landing

4. Luckily, it can never happen $\Psi(t) = \frac{\|p_v^*(t)\|}{m(t)} - p_m^*(t)q = 0$. Otherwise:

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$$\dot{\Psi}(t) = 0 \implies p_r^*(t) \cdot p_v^*(t) = 0 \implies 0 = \frac{d}{dt}\Big(p_r^*(t) \cdot p_v^*(t)\Big) = -\|p_r^*(t)\|^2$$

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In turn, we have discovered optimal controls are bang-bang:

$$u^*(t) = \begin{cases} u_{\min} \frac{p_v^*(t)}{\|p_v^*(t)\|}, & \Psi(t) < 0 \\ u_{\max} \frac{p_v^*(t)}{\|p_v^*(t)\|}, & \Psi(t) > 0 \end{cases}, \quad \Psi(t) = \frac{\|p_v^*(t)\|}{m(t)} - p_m^*(t)q$$

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   b. From $\dot{\Psi}(t) = -\frac{p_r^*(t)\cdot p_v^*(t)}{m(t)\|p_v^*(t)\|}$, it follows that $\dot{\Psi}(t)$ is either always non-negative or it changes sign at maximum once (from - to +).

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   c. In addition, the adjoint system yields

$$\begin{cases} \dot{p}_m^*(t) = \dfrac{p_v^*(t) \cdot u^*(t)}{m(t)^2} \\ p_m^*(t_f) = -p^0 \end{cases} \quad \longrightarrow \quad p_m^*(0) \leq 1 - \int_0^{t_f} \frac{\alpha(t)\|p_v^*(t)\|}{m(t)^2}\, dt$$

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Recall $u^*(t) = \alpha(t)\frac{p_v^*(t)}{\|p_v^*(t)\|}, \quad \alpha(t) \in [u_{\min}, u_{\max}]$

$$\begin{cases} \dot{p}_m^*(t) = \dfrac{p_v^*(t)\cdot u^*(t)}{m(t)^2} \\ p_m^*(t_f) = -p^0 \end{cases} \quad \longrightarrow \quad p_m^*(0) \leq 1 - \int_0^{t_f} \frac{\alpha(t)\|p_v^*(t)\|}{m(t)^2}\,dt$$

Characterize optimal controls for rocket landing

5. We can more carefully characterize such optimal controls.

   a. Indeed, the adjoint system yields

   $$\begin{cases}\dot{p}_v^*(t) = -p_r^*(t) \\ p_v^*(t_f) = \mathfrak{p}_v\end{cases} \quad \longrightarrow \quad \frac{d}{dt}\Big(-p_r^*(t)\cdot p_v^*(t)\Big) = \|p_r^*(t)\|^2 \geq 0$$

   In particular, the function $t \in [0, t_f] \mapsto -p_r^*(t)\cdot p_v^*(t)$ is non-decreasing.

   b. From $\dot{\Psi}(t) = -\frac{p_r^*(t)\cdot p_v^*(t)}{m(t)\|p_v^*(t)\|}$, it follows that $\dot{\Psi}(t)$ is either always non-negative or it changes sign at maximum once (from - to +).

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$$\Psi(0) = \frac{\|p_v^*(0)\|}{m(0)} - p_m^*(0) q > 0$$

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   d. Summary:
      A. $\dot{\Psi}(t)$ always non-negative.
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   e. We conclude:

$$u^*(t) = \begin{cases} u_{\min}, & \Psi(t) < 0 \\ u_{\max}, & \Psi(t) > 0 \end{cases}$$

$$u^*(t) = \begin{cases} u_{\max} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, & t \in [0, t_1] \\ u_{\min} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, & t \in (t_1, t_2], \\ u_{\max} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, & t \in (t_2, t_f] \end{cases} \quad \begin{array}{c} 0 \le t_1 \le t_2 \le t_f \\ w_1,\ w_2 \in \mathbb{R}^3 \end{array}$$

Characterize optimal controls for rocket landing

Minimal fuel consumption reusable rocket landing (with fixed or free final time)

$$\min_u \quad -m(t_f)$$

$$0 < u_{\min} \le \|u(t)\| \le u_{\max}$$

$$\frac{d}{dt}\begin{pmatrix} r \\ v \\ m \end{pmatrix}(t) = \begin{pmatrix} v \\ \frac{u}{m} - g_r \\ -q\|u\| \end{pmatrix}(t)$$

$$(r, v, m)(0) = (r_0, v_0, m_0),$$
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Optimal controls are of the form (under some mild assumptions):

$$u^*(t) = \begin{cases} u_{\max} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, & t \in [0, t_1] \\ u_{\min} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, & t \in (t_1, t_2], \\ u_{\max} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, & t \in (t_2, t_f] \end{cases}$$

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We have reduced the problem to a finite-dimensional optimization problem: just find $0 \leq t_1 \leq t_2 \leq t_f$ and $w_1,\ w_2 \in \mathbb{R}^3$ numerically!!!

Characterize optimal controls for rocket landing

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→ Next lecture!

Course schedule

1. Optimal Control Problems (OCP): review of ordinary differential equations; existence of solutions to OCP.

2. Optimality Conditions for OCP: the Maximum Principle and structure of optimal controls; application to reusable rocket landing.

3. Python Session 1: real-world reusable rocket landing (next class).

4. Python Session 2: training of neural networks through NeuralODE (this application might change depending on the course first outcomes).

5. Final presentation of the results.

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Bring your laptops with Conda installed!!!

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End of lecture 2

Questions?
Otherwise, see you in two weeks!