Advanced optimal control: from reusable rocket landing to efficient training of neural networks

IPSA course 2025

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Laboratoire Signaux & Systèmes





Course schedule

- 1. Optimal Control Problems (OCP): review of ordinary differential equations; existence of solutions to OCP.
- 2. Optimality Conditions for OCP: the Maximum Principle and structure of optimal controls; application to reusable rocket landing.
- 3. Python Session 1: real-world reusable rocket landing.
- 4. Python Session 2: training of neural networks through NeuralODE (this application might change depending on the course first outcomes).
- 5. Final presentation of the results.

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- 5. Final presentation of the results.

 $\sum_{u(\cdot)\in L^\infty} h(x(t_f)) + \int_0^{t_f} f^0(t,x(t),u(t)) \; dt$

(Control) constraints $u(t)\in U\subseteq \mathbb{R}^m, \ \ t\in [0,t_f]$

Dynamics + initial/final conditions

$$egin{cases} x(t) \in \mathbb{R}^n, & t \in [0,t_f], \ \dot{x}(t) = f(t,x(t),u(t)), \ x(0) = x_0, & g(x(t_f)) = 0 \end{cases}$$

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- 2. We studied conditions under which <u>feasible</u> OCP have at least one optimal solution $u^*(\cdot) \in L^{\infty}([0, t_f], U)$, with corresponding optimal trajectory . $x^*: [0, t_f] \to \mathbb{R}^n$

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- 3. <u>Today's topic</u>: characterize optimal solutions $u^*(\cdot) \in L^{\infty}([0, t_f], U)$ to OCP via <u>necessary conditions of optimality</u>.

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Lecture 2 - Characterize the structure of solutions to OCP

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Today's detailed schedule

- 1. Necessary conditions for optimality: the Pontryagin Maximum Principle (PMP).
- 2. Learn how to apply the PMP through simple optimal control problems.
- 3. Characterize optimal controls for the reusable rocket landing problem.

Today's detailed schedule

- 1. Necessary conditions for optimality: the Pontryagin Maximum Principle (PMP).
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Recall OCP:

$$\min_{u(\cdot)\in L^{\infty}} h(x(t_f)) + \int_{0}^{t_f} f^0(t,x(t),u(t)) \ dt \ egin{array}{c} x(t)\in \mathbb{R}^n, \ t\in [0,t_f], \ \dot{x}(t)=f(t,x(t),u(t)), \ u(t)\in U\subseteq \mathbb{R}^m, \ t\in [0,t_f] \end{array} \qquad ext{where} \qquad egin{array}{c} h:\mathbb{R}^n o \mathbb{R} \ \dot{x}(t)=f(t,x(t),u(t)), \ x(0)=x_0, \ g(x(t_f))=0 \end{array} \qquad ext{where} \qquad egin{array}{c} f^0:\mathbb{R}_+ imes \mathbb{R}^n imes \mathbb{R} \ f:\mathbb{R}_+ imes \mathbb{R}^n imes \mathbb{R} \ g:\mathbb{R}^n o \mathbb{R}^\ell \end{array}$$

Key insight: OCP is a (infinite-dimensional) constrained optimization problem.

$$\min_{u\in\mathcal{U}}\ C(u), \ ext{ s.t. } E(u)=0$$

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Let us introduce such conditions for optimality for OCP, and characterize optimal controls!

1. Let us introduce the Hamiltonian:

 $H(t,x,u,p,p^0) riangleq p^ op f(t,x,u)+p^0f^0(t,x,u), \hspace{1em} t\in \mathbb{R}_+, \hspace{1em} x\in \mathbb{R}^n, \hspace{1em} u\in \mathbb{R}^m, \hspace{1em} p\in \mathbb{R}^n, \hspace{1em} p^0\in \mathbb{R}$

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New variables: <u>multipliers</u>!

$\begin{array}{ll} \textbf{Recall OCP:} \\ \underset{u(\cdot)\in L^{\infty}}{\min} \ h(x(t_f)) + \int_{0}^{t_f} f^0(t,x(t),u(t)) \ dt \\ u(t)\in U\subseteq \mathbb{R}^m, \ t\in[0,t_f] \end{array} \qquad \begin{array}{ll} x(t)\in \mathbb{R}^n, \ t\in[0,t_f], \\ \dot{x}(t)=f(t,x(t),u(t)), \\ x(0)=x_0, \ g(x(t_f))=0 \end{array} \qquad \qquad \begin{array}{ll} \textbf{where} & \begin{array}{ll} h:\mathbb{R}^n\to\mathbb{R} \\ f^0:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R} \\ f:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n \\ g:\mathbb{R}^n\to\mathbb{R}^\ell \end{array}$

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2. For any fixed $u(\cdot) \in L^{\infty}([0, t_f], U)$ (and $x_u : [0, t_f] \to \mathbb{R}^n$), and fixed $p^0 \in \mathbb{R}$ and $\mathfrak{p} \in \mathbb{R}^\ell$, the adjoint vector is the continuous curve $p_u : [0, t_f] \to \mathbb{R}^n$ sol. to the adjoint system:

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Key comments:

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Key comments:

a.
$$\nabla_x H = \text{gradient w.r.t. } x$$

$$J_xg = {
m Jacobian w.r.t.} \ x$$

b. <u>Backward linear ODE</u>: unique solution exists

New variables: multipliers!

3. <u>Theorem PMP</u> (necessary conditions for optimality).

Let (u^*, x^*) be an optimal control-trajectory pair solution to OCP. There exists a pair $(p^0, \mathfrak{p}) \in \{-1, 0\} \times \mathbb{R}^{\ell}$ such that the following conditions hold true:

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Let $p^*: [0, t_f] \to \mathbb{R}^n$ be the unique solution the the <u>adjoint system</u>:

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 Evaluated at the optimal pair!

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b. Non-triviality: $(p^*(t),p^0)
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- b. Non-triviality: $(p^*(t),p^0)
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- c. Maximality condition: $u^*(t) \in rgmax_{u \in U} H(t,x^*(t),u,p^*(t),p^0)$

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 Evaluated at the optimal pair!

- b. Non-triviality: $(p^*(t),p^0)
 eq 0$ for all $t\in [0,t_f]$
- c. Maximality condition: $u^*(t) \in rgmax_{u \in U} H(t,x^*(t),u,p^*(t),p^0)$
- d. Final condition (only if t_f is free!!!): $\max_{u \in U} H(t_f, x^*(t_f), u, p^*(t_f), p^0) = 0$

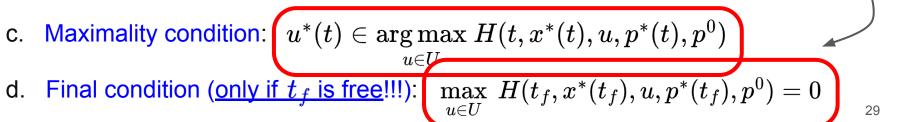
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b. Non-triviality: $(p^*(t), p^0) \neq 0$ for all $t \in [0, t_f]$



of $u^*(\cdot)$.

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- 3. Characterize optimal controls for the reusable rocket landing problem.

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{\substack{u(\cdot)\in L^\infty([0,t_f],[-1,1])\ \dot{z}(t)\,=\,u(t)}} \int_0^{t_f} u(t)^2 \,dt, \hspace{1em} t_f>0 ext{ fixed} \ \begin{cases} \dot{y}(t)\,=\,z(t)\ \dot{z}(t)\,=\,u(t) \end{cases} ext{ with } \begin{cases} y(0)\,=\,1, \hspace{1em} z(0)\,=\,0\ y(t_f)\,=\,z(t_f)\,=\,0 \end{cases}$$

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} \int_0^{t_f} u(t)^2 \, dt, \ \ t_f>0 ext{ fixed} \ \left\{ egin{matrix} \dot{y}(t) = z(t) \ \dot{z}(t) = u(t) \ \end{cases} ext{ with } \left\{ egin{matrix} y(0) = 1, \ z(0) = 0 \ y(t_f) = z(t_f) = 0 \ \end{cases}
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<u>Assuming that there exists an optimal control-trajectory pair</u> (u^*, x^*) , let us apply the PMP and find computationally simpler forms of optimal controls. There are 3 steps:

Minimal energy double integrator - Used in electronic circuit eco-phasing

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<u>Assuming that there exists an optimal control-trajectory pair</u> (u^*, x^*) , let us apply the PMP and find computationally simpler forms of optimal controls. There are 3 steps:

1. Write the Hamiltonian:

$$x riangleq (y,z)\in \mathbb{R}^2, \hspace{0.2cm} u\in \mathbb{R}, \hspace{0.2cm} p riangleq (p_y,p_z)\in \mathbb{R}^2, \hspace{0.2cm} p^0\in \{0,-1\}$$

Minimal energy double integrator - Used in electronic circuit eco-phasing

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Minimal energy double integrator - Used in electronic circuit eco-phasing

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2. Given the Hamiltonian $H(t, x, u, p, p^0) = p_y z + p_z u + p^0 u^2$, there exist a vector $\mathfrak{p} \in \mathbb{R}^2$ and an adjoint $p^* : [0, t_f] \to \mathbb{R}^2$ satisfying the adjoint system:

$$egin{cases} \dot{p}^*(t) = -
abla_x H(t,x^*(t),u^*(t),p^*(t),p^0), \ p^*(t_f) = p^0
abla_x h(x^*(t_f)) + \mathfrak{p}^ op J_x g(x^*(t_f)) \end{split}$$

2. Given the Hamiltonian $H(t, x, u, p, p^0) = p_y z + p_z u + p^0 u^2$, there exist a vector $\mathfrak{p} \in \mathbb{R}^2$ and an adjoint $p^* : [0, t_f] \to \mathbb{R}^2$ satisfying the adjoint system: $\int \dot{p}^*(t) = -\nabla_x H(t, x^*(t), u^*(t), p^*(t), p^0),$

$$iggl\{ p^*(t_f) = p^0
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In particular, we compute

a.

$$egin{aligned} & \left(\dot{p}_y^*(t) \ \dot{p}_z^*(t)
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b. $p^*(t_f) = p^0 \nabla_x h(x^*(t_f)) + \mathfrak{p}^\top J_x g(x^*(t_f)) = (\mathfrak{p}_y, \mathfrak{p}_z) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\mathfrak{p}_y, \mathfrak{p}_z)$

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Thus: $p_y^*(t) = \mathfrak{p}_y, \quad p_z^*(t) = \mathfrak{p}_z + \mathfrak{p}_y(t_f - t), \quad t \in [0, t_f]$

3. Given the Hamiltonian $\, H(t,x,u,p,p^0) = p_y z + p_z u + p^0 u^2$ and the adjoint

 $p_y^*(t) = \mathfrak{p}_y, \;\; p_z^*(t) = \mathfrak{p}_z + \mathfrak{p}_y(t_f - t)$, solve the maximality condition

$$u^*(t)\in rgmax_{u\in U} H(t,x^*(t),u,p^*(t),p^0)$$

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and the final condition (only if t_f is free!)

$$\max_{u \in U} \; H(t_f, x^*(t_f), u, p^*(t_f), p^0) = 0$$

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3. Given the Hamiltonian $H(t, x, u, p, p^0) = p_y z + p_z u + p^0 u^2$ and the adjoint $p_y^*(t) = \mathfrak{p}_y, \quad p_z^*(t) = \mathfrak{p}_z + \mathfrak{p}_y(t_f - t)$, solve the maximality condition $u^*(t) \in \operatorname*{arg\,max}_{u \in U} H(t, x^*(t), u, p^*(t), p^0)$ and the final condition (only if t_f is free!) $\max_{u \in U} H(t_f, x^*(t_f), u, p^*(t_f), p^0) = 0$ Final time is fixed in this example!

Need to find solutions to the <u>finite-dimensional optimization</u>:

$$rgmax_{u\in U} H(t,x^*(t),u,p^*(t),p^0) = rgmax_{u\in [-1,1]} (p_z^*(t)u+p^0u^2)$$

where $p^0\in\{0,-1\}$

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44

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Need to find solutions to the *finite-dimensional optimization*:

$$\begin{array}{l} \operatorname*{arg\,max}_{u\in U} H(t,x^*(t),u,p^*(t),p^0) = \operatorname*{arg\,max}_{u\in [-1,1]} (p_z^*(t)u+p^0u^2) \\ \\ \text{where} \quad p^0 \in \{0,-1\} \\ \\ \begin{array}{c} \text{Solutions depend} \\ \text{on the value of } p^0 \,! \end{array} \right) \\ \end{array}$$

Let us solve: $rgmax_{u \in U} H(t, x^*(t), u, p^*(t), p^0) = rgmax_{u \in [-1,1]} (p_z^*(t)u + p^0u^2)$

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a. Case $p^0=-1$

Maximizing a parabola:

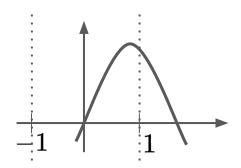
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Maximizing a parabola: maximum at the vertex if in [-1,1]

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Let us solve: $rgmax_{u \in U} H(t, x^*(t), u, p^*(t), p^0) = rgmax_{u \in [-1,1]} (p_z^*(t)u + p^0u^2)$

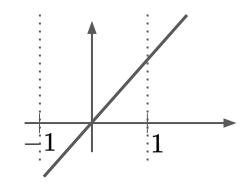
b. Case $p^0=0$

 $\text{Let us solve:} \quad \operatorname*{arg\,max}_{u \in U} H(t, x^*(t), u, p^*(t), p^0) = \operatorname*{arg\,max}_{u \in [-1,1]} (p_z^*(t)u + p^0u^2)$

b. Case $p^0=0$

Maximizing a straight line in [-1,1]

$$u^*(t) = rgmax_{u\in [-1,1]} p_z^*(t) u$$

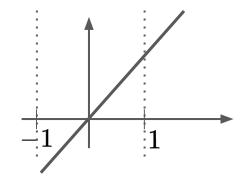


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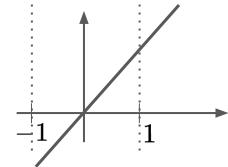


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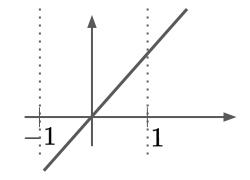


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Luckily, $p_z^*(t) = 0$ and $p^0 = 0$ <u>never happens</u>.

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Maximizing a straight line in [-1,1]

$$u^{*}(t) = \underset{u \in [-1,1]}{\arg \max} p_{z}^{*}(t)u = \begin{cases} -1, \text{ if } p_{z}^{*}(t) < 0\\ 1, \text{ if } p_{z}^{*}(t) > 0\\ ???, \text{ if } p_{z}^{*}(t) = 0 \end{cases}$$
What if $p_{z}^{*}(t) = 0$???

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Minimal energy double integrator - Used in electronic circuit eco-phasing

$$egin{aligned} &\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} &\int_0^{t_f} u(t)^2 \ dt, &t_f>0 ext{ fixed} \ &iggl\{ \dot{y}(t)=z(t) \ \dot{z}(t)=u(t) & ext{with } &iggl\{ y(0)=1, \ z(0)=0 \ y(t_f)=z(t_f)=0 \end{aligned}$$

We have finally showed that optimal controls for this OCP can only take two forms:

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{\substack{u(\cdot)\in L^\infty([0,t_f],[-1,1])\ \dot{z}(t)\,=\,u(t)}} \int_0^{t_f} u(t)^2 \,dt, \hspace{1em} t_f>0 ext{ fixed} \ \begin{cases} \dot{y}(t)\,=\,z(t)\ \dot{z}(t)\,=\,u(t) \end{cases} ext{ with } \begin{cases} y(0)\,=\,1, \hspace{1em} z(0)\,=\,0\ y(t_f)\,=\,z(t_f)\,=\,0 \end{cases}$$

We have finally showed that optimal controls for this OCP can only take two forms: By denoting $a \triangleq t_f - \frac{2 - \mathfrak{p}_z}{\mathfrak{p}_y}$, $b \triangleq t_f + \frac{2 + \mathfrak{p}_z}{\mathfrak{p}_y}$, and $c \triangleq t_f + \frac{\mathfrak{p}_z}{\mathfrak{p}_y}$, we have

Minimal energy double integrator - Used in electronic circuit eco-phasing

$$\min_{u(\cdot)\in L^\infty([0,t_f],[-1,1])} \int_0^{t_f} u(t)^2 \ dt, \ \ t_f>0 \ ext{fixed} \ \left\{ egin{matrix} \dot{y}(t) = z(t) \ \dot{z}(t) = u(t) \end{matrix}
ight. ext{with} \ \ \left\{ egin{matrix} y(0) = 1, \ z(0) = 0 \ y(t_f) = z(t_f) = 0 \end{matrix}
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We have finally showed that optimal controls for this OCP can only take two forms: By denoting $a \triangleq t_f - \frac{2 - \mathfrak{p}_z}{\mathfrak{p}_y}$, $b \triangleq t_f + \frac{2 + \mathfrak{p}_z}{\mathfrak{p}_y}$, and $c \triangleq t_f + \frac{\mathfrak{p}_z}{\mathfrak{p}_y}$, we have $u^*(t) = \begin{cases} \frac{2}{a-b}t - \frac{a+b}{a-b}, & \text{if } t \in [a,b] \\ -1, & \text{if } t > b \\ 1, & \text{if } t < a \end{cases}$

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61

Existence of solutions to OCP - Examples

Minimal energy reusable rocket landing

 $\min_{u\in L^\infty([0,t_f],U)} \;\; \int_0^{t_f} \|u(t)\|^2 \; dt, \;\;\; t_f>0 \; ext{fixed}$

$$U riangleq \{u \in \mathbb{R}^3: \ \|u\| \leq u_{ ext{max}}\} \subseteq \mathbb{R}^3 \qquad rac{d}{dt} inom{r}{v}(t) = inom{v(t)}{ar{m}} - g_rigg) \qquad inom{(r,v)(0)}{\{r,v)(0)} = (r_0,v_0) \ (r,v)(t_f) = (0,0)$$

<u>It's your turn</u>: By assuming that this OCP has at least one optimal control-trajectory pair (u^*, x^*) , apply the PMP and show $u^*(\cdot)$ is of the form (20/30 minutes):

Existence of solutions to OCP - Examples

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$$u^*(t) = egin{cases} \mathfrak{f}(t), ext{ if } \|\mathfrak{f}(t)\| \leq u_{ ext{max}} \ u^*(t) = egin{cases} \mathfrak{f}(t), ext{ if } \|\mathfrak{f}(t)\| \leq u_{ ext{max}} \ rac{\mathfrak{f}(t)}{\|\mathfrak{f}(t)\|}, ext{ if } \|\mathfrak{f}(t)\| > u_{ ext{max}} \end{cases}$$

$$\underline{\mathsf{DR}} \qquad u^*(t) = u_{\max} rac{\mathfrak{f}(t)}{\|\mathfrak{f}(t)\|}$$

Solution: See the blackboard.

Today's detailed schedule

- 1. Necessary conditions for optimality: the Pontryagin Maximum Principle (PMP).
- 2. Learn how to apply the PMP through simple optimal control problems.
- 3. Characterize optimal controls for the reusable rocket landing problem.

Minimal fuel consumption reusable rocket landing (with fixed or free final time)

$$egin{array}{ll} \displaystyle \min_u & -m(t_f) & \displaystyle rac{d}{dt} egin{pmatrix} r \ v \ m \end{pmatrix}(t) = egin{pmatrix} v \ rac{T_{ ext{max}u}}{m} - g_r \ -q \|u\| \end{pmatrix}(t) & (r,v,m)(0) = (r_0,v_0,m_0), \ (r,v)(t_f) = (0,0) \end{array}$$

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Let us apply the PMP and characterize optimal control-trajectory tuples $(u^*, x^*)!$

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1-2. For the Hamiltonian and the adjoint system we have (see the blackboard): $x \triangleq (r, v, m) \in \mathbb{R}^7, \ u \in \mathbb{R}^3, \ p \triangleq (p_r, p_v, p_m) \in \mathbb{R}^7, \ p^0 \in \{0, -1\}$

Minimal fuel consumption reusable rocket landing (with fixed or free final time)

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$$\mathfrak{p}_r \qquad \Big(p_v^*(t_f) = \mathfrak{p}_v \qquad \qquad \Big(p_m^*(t_f) = -p^0 \Big)^{71} \Big)$$

3. The Hamiltonian $H(t,x,u,p,p^0) = p_r^ op v + p_v^ op \left(rac{u}{m} - g_r
ight) - p_m q \|u\|$ yields solving

$$u^*(t) = rgmax_{0 < u_{\min} \leq \|u\| \leq u_{\max}} \left(rac{p_v^*(t) \cdot u}{m(t)} - p_m^*(t) q \|u\|
ight)$$

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ight) \ &= rgmax_{lpha \in [u_{\min}, u_{\max}], \ n \in S^2} lpha \left(rac{p_v^*(t) \cdot n}{m(t)} - p_m^*(t) q
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*(1)

* / 1)

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In contradiction with our assumption $\, p_r^*(t)
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In contradiction with our assumption $\, p_r^*(t)
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$$\begin{array}{l} \text{In turn, we have} \\ \text{discovered optimal} \\ \text{controls are bang-bang:} \quad u^*(t) = \begin{cases} u_{\min} \frac{p_v^*(t)}{\|p_v^*(t)\|}, \ \Psi(t) < 0 \\ u_{\max} \frac{p_v^*(t)}{\|p_v^*(t)\|}, \ \Psi(t) > 0 \end{cases}, \quad \Psi(t) = \frac{\|p_v^*(t)\|}{m(t)} - p_m^*(t)q \\ \end{array}$$

5. We can more carefully characterize such optimal controls.

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In particular, the function $t\in [0,t_f]\mapsto -p_r^*(t)\cdot p_v^*(t)$ is non-decreasing.

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b. From $\dot{\Psi}(t) = -\frac{p_r^*(t) \cdot p_v^*(t)}{m(t) \|p_v^*(t)\|}$, it follows that $\dot{\Psi}(t)$ is either always non-negative or it changes sign at maximum once (from - to +).

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- We can more carefully characterize such optimal controls. 5.
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b

$$\begin{cases} \dot{p}_v^*(t) = -p_r^*(t) \\ p_v^*(t_f) = \mathfrak{p}_v \\ \text{In particular, the function} \quad t \in [0, t_f] \mapsto -p_r^*(t) \cdot p_v^*(t) \text{ is non-decreasing.} \\ \end{cases}$$
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c. In addition, the adjoint system yields Recall $u^*(t) = \alpha(t) \frac{p_v^*(t)}{|| p_v^*(t) ||}$, $\alpha(t) \in [u_{\min}, u_{\max}]$

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$$p_m^*(0) \leq 1 - \int_0^{t_f} \frac{\alpha(t) \| p_v^*(t) \|}{m(t)^2} dt < 0 \text{ for long } t_f \end{cases}$$

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- 5. We can more carefully characterize such optimal controls.
 - d. Summary:
 - A. $\dot{\Psi}(t)$ always non-negative.
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$$\begin{array}{ll} \text{e. We conclude:} \\ u^*(t) = \begin{cases} u_{\min}, \ \Psi(t) < 0 \\ u_{\max}, \ \Psi(t) > 0 \end{cases} \\ u^*(t) = \begin{cases} u_{\max}, \ \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, \ t \in [0, t_1] \\ u_{\min} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, \ t \in (t_1, t_2], \\ u_{\max} \frac{w_1 + w_2 t}{\|w_1 + w_2 t\|}, \ t \in (t_2, t_f] \end{cases} \\ \begin{array}{ll} 0 \le t_1 \le t_2 \le t_f \\ w_1, \ w_2 \in \mathbb{R}^3 \\ y_4 \end{cases} \\ \end{array}$$

Minimal fuel consumption reusable rocket landing (with fixed or free final time)

$$egin{array}{ll} \min_u & -m(t_f) & & \ rac{d}{dt}inom{r}{v} & \left(t
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Optimal controls are of the form (under some mild assumptions):

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Minimal fuel consumption reusable rocket landing (with fixed or free final time)

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We have reduced the problem to a finite-dimensional optimization problem: just find $0 \le t_1 \le t_2 \le t_f$ and $w_1, w_2 \in \mathbb{R}^3$ numerically!!!

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Course schedule

- 1. Optimal Control Problems (OCP): review of ordinary differential equations; existence of solutions to OCP.
- 2. Optimality Conditions for OCP: the Maximum Principle and structure of optimal controls; application to reusable rocket landing.
- 3. Python Session 1: real-world reusable rocket landing (next class).
- 4. Python Session 2: training of neural networks through NeuralODE (this application might change depending on the course first outcomes).
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Bring your

laptops with

End of lecture 2

Questions? Otherwise, see you in two weeks!